

SMART: A Stochastic Multiscale Model for the Analysis of Energy Resources, Technology and Policy

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We address the problem of modeling energy resource allocation, including dispatch, storage and the long-term investments in new technologies, capturing different sources of uncertainty such as energy from wind, demands, prices and rainfall. We also wish to model long-term investment decisions in the presence of uncertainty. Accurately modeling the value of all investments such as wind and solar requires handling fine-grained temporal variability and uncertainty in wind and solar in the presence of storage. We propose a modeling and algorithmic strategy based on the framework of approximate dynamic programming (ADP) that can model these problems at hourly time increments over an entire year, or over several decades. We demonstrate the methodology using both spatially aggregate and disaggregate representations of energy supply and demand. This paper describes initial proof of concept experiments for an ADP-based model, called SMART, by describing the modeling and algorithmic strategy, and providing comparisons against a deterministic benchmark as well as initial experiments on stochastic datasets.

Key words: artificial intelligence; simulation; statistical analysis; analysis of algorithms; queues

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1. Introduction

Over the next few decades, the United States needs to transition to an energy system with much lower carbon emissions and reduced dependence on unreliable supplies. This process requires significant investment in R&D and new infrastructure. The investments have to be made in the presence of multiple sources of uncertainty that could influence the future energy system, such as the mix of primary energy (nuclear, renewable, sequestered fossil), and the balance of energy carriers including electricity, hydrogen, alcohol and synthetic fuels, and end-use strategies (level of demand, demand response). A major component will be renewable energy that depend on intermittent sources from wind and solar. The volatility of these sources can be mitigated with different types of storage such as batteries and pumped hydro.

The dynamics of our system are driven by exogenous factors (wind, solar, climate, technology, prices, supplies and demands) and market decisions (how much energy should be

produced by each source, how much of this energy should be used for each type of demand, how much energy should be stored, and how much capacity should be added or retired each year in response to changing conditions). To model investment decisions and government policies, the model has to extend over a multi-decade horizon. To accurately represent the economics of energy technologies, our model has to capture the hourly fluctuations of wind, daily solar cycles, seasonal climate changes, and annual changes in energy technologies. Finally, it is widely understood that a key enabling technology for intermittent energy is storage, which complicates the design of algorithms by coupling all the time periods together.

Past models have represented market investment decisions by assuming they can be reasonably approximated as optimal solutions to deterministic linear programs. These models have been effective at capturing fundamental differences in supply and demand, where the most famous example is the ground breaking policy conclusions from the PIES model Hogan (1975). In recent years, there has been growing interest in the use of models to guide investment and policy decisions in new technologies such as wind and solar, and to guide government policies for carbon taxes and research and development to meet goals such as 20 percent energy from renewables by 2030. It has long been recognized, however, that models that represent intermittent sources such as wind and solar overstate their value if they use averages (wind turbines are dramatically more productive if wind blows at a constant speed). The problem with intermittent sources is that they are not dispatchable (directly controllable in real time), requiring investments in other energy technologies to make up for low periods, even if the average production is very high (see Lamont (2008) for a thorough discussion of these issues). Energy policy researchers have found that using time increments longer than an hour begin to introduce significant errors. Storage can overcome the limitations of intermittent sources, but representing storage imposes additional demands on a model.

This paper extends a deterministic energy planning model called META*Net (see Lamont (1997) and Lamont (2008)) developed at Lawrence Livermore National Laboratory which has been used in two forms: a model which captures hourly variations over a year, and a model which extends for several decades in increments of one to five years. META*Net is an aggregate model, which captures approximately 13 different sources of energy to satisfy 9 types of demand, spanning different types of electric power as well as different energy forms for industrial and residential purposes. Aggregate planning models such as these stand between simple models which analyze the economics of different types of energy generation in isolation, and detailed network models used in operational planning for utilities and grid operators.

Separate from the need to handle fine-grained time scales is the need to capture different types of uncertainty. This ranges from uncertainty about wind (wind operators often have to commit to a certain level of production a day in advance), to uncertainties about policies (imposition of a carbon tax), technology (will we be able to sequester carbon), prices (what will be the price of oil in five years) and climate (how will global warming affect snow accumulation and rainfall). These uncertainties can impact the risks incurred in an investment, significantly reducing the appeal of technologies that look attractive if the future unfolds as expected.

Our goal is to develop a model which is stochastic, handling different sources of uncertainty, and multiscale, handling different levels of spatial and temporal granularity. We also

need to incorporate storage, which has the effect of linking different time periods together. Finally, we are interested in near-optimal policies, which we achieve using approximate dynamic programming. This dimension provides a sharp contrast with other papers which may use a stochastic *model*, but do not account for uncertainty when making a decision. We use deterministic benchmarks to demonstrate the quality of our solutions, but we are limited to qualitative evaluations to assess our ability to properly reflect uncertainty in our policies.

These goals represent what we feel that we are able to achieve in this paper, but there are other goals to which we aspire. We are able to handle fine-grained sources of uncertainty (wind, rainfall, prices, demand), but we would like to handle coarse-grained forms of uncertainty which include changes in tax policy or breakthroughs in technology. We can handle spatially disaggregate supply and demand patterns, but we cannot model individual power plants which requires the introduction of integer variables governing when a plant is in use or not.

This paper describes an early implementation of “SMART” (Stochastic Multiscale model for the Analysis of energy Resources, Technology and policy). SMART uses a control-theoretic framework to step forward through time in hourly increments over a multidecade horizon, which also makes it possible to model energy storage. Approximate dynamic programming is used to produce policies that properly model the flow of information when making decisions. Although designed to handle stochastic problems, SMART can be applied to deterministic problems, making it possible to compare the performance of the model to a deterministic linear programming model, when such a model can be solved. We show that SMART closely matches the optimal solution from a deterministic linear program, which provides an important benchmark for our ADP algorithm. We also show that it exhibits robust behaviors when applied to a stochastic problem. To the best of our knowledge, this is the first stochastic, multiscale model that scales to handle hundreds of thousands of time periods, while producing solutions that closely match the results of optimal solutions for deterministic formulations.

SMART can be used for both long-term policy studies, as well as economic analyses of portfolios of energy production and storage technologies. It can be used to simulate energy investment decisions, under the assumption that a cost-minimizing model mimics market behavior, or as a normative model to determine how to reach specific energy targets. It can be used to quantify risk, and to study the impact of different energy investment strategies. It can also provide accurate estimates of the value of intermittent energy sources and the value of storage using realistic assumptions about our ability to forecast energy supply.

This paper is organized as follows. Section 2 reviews the literature for models on energy policy, and briefly touches on the literature for energy storage and wind. Section 3 provides a sketch of our modeling and algorithmic strategy. An outline of the mathematical model is described in section 4, but the full model is contained in the online appendix, which contains a number of important equations to which the paper refers. Our model captures both the physical dynamics and the flow of information and decisions, but does not provide a method for making decisions. Section 5 provides two optimization formulations to drive the process of making decisions: a deterministic linear programming model (which we later use as a benchmark), and a stochastic optimization model. Section 6 describes an algorithmic strategy for solving the stochastic optimization problem based on approximate dynamic programming. In section 7, we describe benchmarking experiments which compare the solution

obtained using approximate dynamic programming to the optimal solution produced by a linear programming algorithm using a deterministic model. Experiments on stochastic applications are also reported to demonstrate specific behaviors that we would expect from a nonanticipative decision rule. Section 8 concludes the paper.

2. Literature review

There are several bodies of literature that are relevant to our research: deterministic energy investment models, stochastic optimization models in energy, models for energy storage, and studies of the value of energy from wind.

Following the original PIES model (Hogan (1975)), a series of deterministic optimization models have been developed to model energy investment decisions over multidecade horizons. These include the National Energy Modeling System (NEMS, NEMS (2003)) that takes as input demands and costs of various energy resources from a family of models. The MARKet ALlocation (MARKAL) model is a deterministic, optimization-based system that integrates the supply and end-use sectors of an economy (see Fishbone and Abilock (1981) and Loulou et al. (2004)). MARKAL minimizes the discounted total cost of the system which includes investment, maintenance and operations costs of the technologies and the purchase costs of fuels, and also accounts for revenue generated. META*Net (see Lamont (1997), Lamont (2008)), developed at the Lawrence Livermore National Laboratory, combines the techniques used in NEMS as well as MARKAL in that it performs market equilibration in a linear programming-based model.

The early stochastic energy investment models can be divided into two categories. We refer to the first category as partially stochastic models, such as the stochastic version of MARKAL Johnson et al. (2006) developed by the Environmental Protection Agency. In this version of MARKAL, a “stochastic wrapper” is used, wherein random parameters are sampled from probability distributions, after which a deterministic optimization problem is solved. This process is repeated a number of times, allowing the outputs to be analyzed statistically. This means that for each run on a sample realization, decisions are allowed to see events in the future. Kanudia and Loulou (1998) presents a version of MARKAL which explicitly incorporates multiple scenarios and includes as many replications of the MARKAL variables as the number of scenarios considered. The optimization is done by minimizing total expected costs over all the scenarios, but where the solution for each scenario is independent of the other scenarios. Again, this strategy allows a decision to see outcomes in the future for a particular scenario.

Another example of a partially stochastic model is the stochastic version of MESSAGE (Model for Energy Supply Strategy Alternatives and their General Environmental Impact) from the International Institute for Applied Systems Analysis (IIASA). The conventional MESSAGE model Messner and Strubegger (1995) uses point estimates to forecast technology parameters. The stochastic MESSAGE model Messner et al. (1996) aims to reduce the risk of underestimating the total investment cost as a stochastic optimization problem, using a sufficiently large number of sample realizations of the investment costs.

The second category of stochastic models uses the framework of stochastic programming to handle the modeling of random information and decisions (see Wallace and Fleten (2003))

for an excellent introduction to these models). Stochastic programming models use the notion of scenario trees to represent uncertain information. The challenge with this framework is that the size of a scenario tree grows exponentially with the number of time periods, requiring the use of aggregation strategies such as large time steps and small numbers of scenarios per time period. See Birge and Louveaux (1997) for an-depth treatment of this approach, and Hoyland and Wallace (2001), Grawe-Kuska et al. (2003) and Kaut and Wallace (2003) for research on generating scenario trees. A competing strategy uses dual information to generate cuts to approximate the impact of decisions now on the future. This was originally introduced as the L-shaped method Van Slyke and Wets (1969), and later adapted (for two-stage problems) using a Monte Carlo-based framework as stochastic decomposition in Higle and Sen (1991) (see Higle and Sen (1996) for a more in-depth treatment).

One limitation of models based on stochastic programming is that the scenarios (outcomes of random information) cannot depend on the sequence of decisions. For example, if decisions lead to energy shortages, prices may spike. Stochastic programming requires that scenarios be generated in advance, and this limits their ability to capture the interaction between decisions and exogenous events. However, stochastic programming is able to capture complex, history dependent processes such as the timing of a carbon tax or breakthroughs in technology.

The last class of stochastic energy investment models can be viewed as pure simulation models. Perhaps the most visible example of this class is the Stochastic Energy Deployment Systems (SEDS) (see <http://seds.nrel.gov/>), which uses classical Monte Carlo simulation to step forward through time. Let ω^n represent the sample realization of all random variables for the n^{th} iteration of the model, and let $c_{te}(\omega^n)$ be the cost of energy technology $e \in \mathcal{E}$ in year t , while following sample path ω^n . SEDS makes additional investments, given by x_{te}^n , according to the expression

$$x_{te}^n = \beta_t^D \frac{\exp -\beta^s c_{te}(\omega^n)}{\sum_{e' \in \mathcal{E}} \exp -\beta^s c_{te'}(\omega^n)} \quad (1)$$

where β_t^D is a set of parameters (over time) which allow investments to be scaled to demand, and β^s is a scaling coefficient that controls the degree to which investments are directed toward lower cost technologies. There is no attempt to formally optimize energy investments. A major goal of the model is to provide fast estimates of the effects of policy changes in a fairly transparent way. The model works in yearly (or even multiyear) increments, so there is no attempt to capture fine-grained variations in wind and solar energy or demand.

There is a separate literature on the use of storage, largely divided between the management of water reservoirs and other forms of energy storage. The literature on water reservoir management is fairly extensive (see Foufoula-Georgiou and Kitanidis (1988), Lamond and Sobel (1995), Archibald et al. (1997) and Cervellera et al. (2005) for a sample). Algorithmic research addressing the problem of optimizing flow controls under uncertainty has progressed along two lines. The first uses dynamic programming, typically resorting to some sort of numerical or approximate strategy to handle the curse of dimensionality that arises with multiple reservoirs (see Nandalal and Bogardi (2007) and the references cited there). A major breakthrough in this approach involves approximating the impact of decisions now on the future using cuts generated from the dual of the linear program in the next time period. Pereira and Pinto (1991) is the first to use this strategy in an energy setting under the

name of stochastic, dual dynamic programming (SDDP), modeling a large, multi-reservoir hydroelectric system. A separate modeling and algorithmic strategy within the stochastic programming community uses the concept of scenario trees, where sample paths of exogenous information are generated, retaining the entire history at each node in the tree. Jacobs et al. (1995) and De Ladurantaye et al. (2007) give nice illustrations of this strategy in a hydroelectric setting. Heuristic approaches to reservoir management are described in Johnson et al. (1991), Edmunds and Bard (1990b) and Edmunds and Bard (1990a).

A separate literature has evolved around the modeling of storage devices to handle either structural or random variations between our ability to generate power and the demand for power. The recent paper Brunetto and Tina (2007) uses a deterministic model to solve the problem of day-ahead electricity markets, completely ignoring the high level of uncertainty that arises in this setting. Castronuovo and Lopes (2004) describes wind as a stochastic process, but then represents only the variability of wind, and not its uncertainty. Korpaas et al. (2004) and Paatero and Lund (2005) also use deterministic models to study storage for wind. Garcia-Gonzalez et al. (2008) and Brown et al. (2008) use the idea of stochastic wind scenarios, but allows decisions to “see” the wind in the future within a particular scenario (violating what are known as nonanticipativity conditions).

There is a body of literature that is working to understand the impact of large amounts of wind on the power grid. Sioshansi and Short (2009) uses a two-day deterministic forecast of wind in a rolling horizon model to simulate decisions. Sioshansi (2010) uses a deterministic unit commitment model (requiring a deterministic forecast of wind) to study the impact of real-time pricing on the use of energy from wind. Holttinen et al. (2011) undertakes a careful study of the effect of large quantities of wind (also see the summary in Holttinen et al. (2009)). However, no formal mathematical models are given (this includes the Ph.D. dissertation on which this work is based, given in Holttinen (2004)), and it does not appear that the study uses a policy that explicitly accounts for the uncertainty in wind. Separate from the issue of handling uncertainty is the challenge of capturing hourly variations in long-term models. A modeling strategy that avoids capturing every hour over a year (8,760 time periods) involves simulating only a sample of different time periods (capturing, for example, different times of year and different times of day), but such an approach makes it impossible to accurately model storage (whether it is hydro, compressed air or batteries) which requires the ability to step forward across all time periods to calculate storage levels.

3. Strategy for a stochastic, multiscale model

The goal of our research is to develop a model that allows us to make short-term decisions about electric dispatch, energy resource allocation and storage over the short term (one year), as well as long-term investment decisions in the presence of different sources of uncertainty. We are interested in questions such as the impact of increasing the fraction of energy generated from wind and solar, and in obtaining an accurate estimate of the marginal value of wind and solar in the presence of storage. At the same time, we are not looking to address decisions such as where to locate a wind farm, or whether a particular coal or natural gas facility should be used at a particular point in time. These details are important for operational problems such as unit commitment, or specific investments such as additions to the

grid or energy facilities at specific locations. Our model does not represent individual energy generators, since the resulting model would have to include thousands of integer variables per time period. Even without these details, we face a difficult algorithmic challenge.

We anticipate running our model over two time frames. For studies of the effect of the impact of storage on the marginal value of wind and solar, it makes sense to run the model in hourly increments over a year. For studies of the impact of tax subsidies, commodity prices and breakthroughs in technology, we will want to extend the horizon over several decades, while retaining the ability to model variations in wind, solar and demand in hourly increments. For problems where we are only interested in modeling a single year, we may be interested in a spatially disaggregate model, while for long-term investment decisions, a spatially aggregate model may be sufficient.

We use the modeling and algorithmic framework of approximate dynamic programming where dispatch, storage and investment decisions at hour h in year t are represented by the vector x_{th} , which is determined by a decision function (or *policy*, in the language of dynamic programming) $X^\pi(S_{th})$ which depends on the system state variable S_{th} . S_{th} is designed to include only the information available at time (t, h) , and as a result decisions are not allowed to anticipate events in the future.

Our strategy involves stepping forward through time as a simulation model would, but repeating the process while learning from each iteration. Once each year (corresponding to $h = 0$), we make investment decisions for new energy capacity that will arrive in a future year. Then, for the remainder of the year we step forward solving hourly dispatch problems which determines how much energy to produce from each source to satisfy each type of demand. Energy demands are comprised of residential, commercial and industrial demands for electricity and natural gas (residential uses include cooking and heating), as well as freight transportation and aircraft demands for electricity, hydrogen and hydrocarbon fuels. Primary energy resources include coal, petroleum, natural gas, wind, solar, temperature gradients (geothermal, ocean thermal), nuclear, biomass and tides. These undergo conversion processes in plants to generate electricity, hydrocarbon fuels or hydrogen, each of which is then channeled to the appropriate sources of demand. While most of the conversion plants that handle carbon-based energy sources release CO_2 to the atmosphere, some technologies are able to sequester CO_2 . The dispatch problem also includes decisions of how much to store, and how much to convert to different types of energy (for example, natural gas can be used to generate electricity which in turn can be used to convert biomass to fuel or to compress hydrogen).

We demonstrate SMART using two networks. The first is a spatially aggregate model, a portion of which is illustrated in figure 1, which represents 13 different types of energy supplies, and 9 types of energy demand. If there is a shortage of energy supplied from these resources, there also exists the option of importing electricity. Each node in the network represents an aggregated version of a facility in a typical energy network. Resource nodes provide the supply of raw resource (such as coal, natural gas, wind, solar, biomass, nuclear) at a certain cost. Conversion nodes take as inputs the raw resource and some type of power needed to generate more power in some form such as electricity, natural gas and transportation fuel. The flows out of each of these nodes are constrained by the capacity available at that node, which is controlled by investment decisions. Demand nodes capture the demand for one of the various types of power. Market nodes accumulate energy from

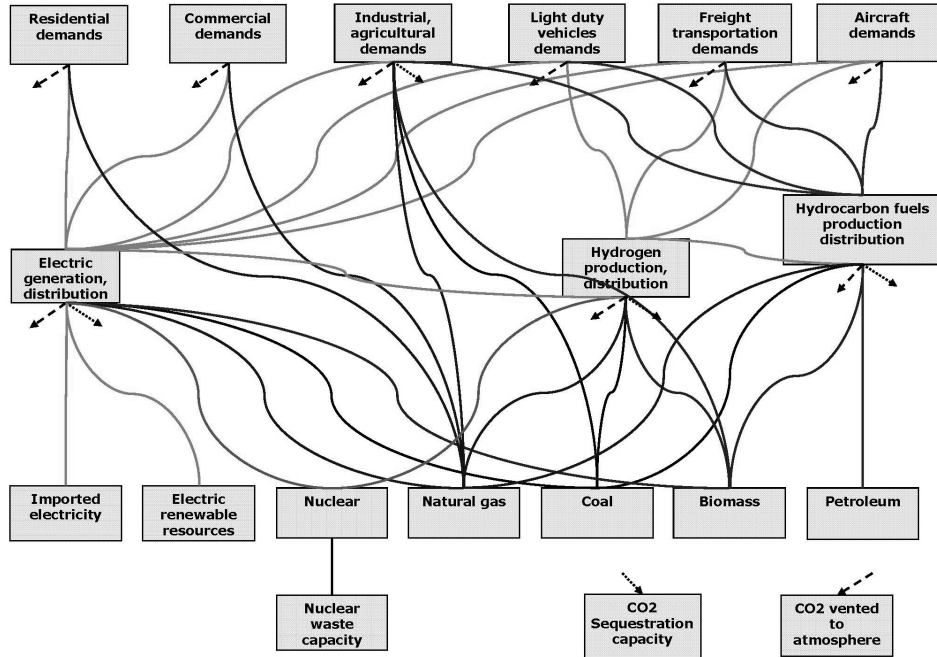


Figure 1: Reduced version of the hourly dispatch problem with electricity demands only.

different sources to be distributed to the various types of demand (these can be sectors or locations). The network also captures interactions such as the need to produce more electricity if it is needed to produce hydrogen or ethanol. We may also add other nodes in this illustration to capture costs involved in the transportation of the energy resource (for example, from the source to the conversion plant) and the distribution of the final output (for example, electricity through transmission lines).

The second network model, used in a study of intermittent energy by Tagher (2010), captures the generation of electricity from coal, nuclear, natural gas and wind from each of the 48 contiguous states, to satisfy different demand sectors in each of the 48 states plus the District of Columbia. Unlike our aggregate network model, this model only considers electricity. The model did not have an explicit representation of the grid, but state to state distances were used to compute transportation costs and transmission losses, using individual links for state to state transmission. This representation of power flows, which captures point to point movements without explicitly modeling the grid, a common approximation made in policy studies. The linear program for a single time period had approximately 40,000 rows and columns.

We do not model individual energy generators, and nor do we model the grid itself. This level of detail is required for operational models, and for certain types of regional planning models which require the identification of bottlenecks in the grid. Including this detail in a model that spans 20 years in hourly increments is a goal of the energy modeling community, but we are not ready to claim that we can handle a model with this granularity. We include transmission losses at an aggregate level, and we can model limits on interregional flows (as the CAISO model does). The model can be used to compare the economics of wind versus shale gas, coal and nuclear under different taxation policies and targets for renewables.

Our modeling and algorithmic strategy involves simulating our way through the time periods, solving small optimization problems to handle dispatch and storage decisions (each hour) and investment decisions (each year). As we step forward through time, we use Monte Carlo simulation to sample realizations from any random variables. Our challenge is designing a decision function $X^\pi(S_{th})$ that produces economically rational decisions, without violating restrictions on the availability of information.

We begin by providing an outline of the model of the problem in section 4 (the complete model is given in the online supplement, section A.1), followed by both deterministic and stochastic optimization formulations in section 5. The mathematical model covers both the aggregate and disaggregate networks.

4. Outline of the energy model

We let $t \in \mathcal{T}$ index years, and $0 \leq h < 8760$ index the hours within a year. Our model consists of five fundamental components:

- S_{th} = The state of the system at time (t, h) .
- x_{th} = The vector of decisions at time (t, h) .
- W_{th} = The vector of random variables representing all the new information that just became available in the hour prior to (t, h) .
- $S^M(S_{th}, x_{th}, W_{t,h+1})$ = The transition function (or system model) that governs the evolution from state S_{th} to $S_{t,h+1}$ (or from $S_{t,H}$ to $S_{t+1,0}$) given decision x_{th} and new information $W_{t,h+1}$.
- C_{th} = The contribution function capturing rewards earned during the hour preceding (t, h) .

The state variable consists of the current status of our energy investments R_{th} , energy held in different forms of storage y_{th} , energy demands D_{th} , rainfall levels p_{th} , and a generic “state of the world” vector ρ_{th} that captures information about technology, climate, government policy, and the availability of intermittent energy such as wind and solar.

We let the attributes of energy resources or storage be represented by the vector a . In our aggregate model, a would represent the type of technology, its location and age. For the disaggregate model, we add the attribute “state” to capture geography. We let \mathcal{A} be the set of all possible attribute vectors, and we let \mathcal{A}^{Conv} and \mathcal{A}^{Stor} represent the sets of attribute vectors that correspond to conversion and storage resources, respectively. We let R_{tha} be the level of investment in a technology with attribute $a \in \mathcal{A}$ (for example, this could be the megawatt-hours of generating capacity of a wind farm), and let $R_{th} = (R_{tha})_{a \in \mathcal{A}}$ be the vector of energy investments. The vector $y_{th} = (y_{tha})_{a \in \mathcal{A}^{Stor}}$ captures the amount of energy in each type of storage (note that our numerical work includes only hydro storage). It is easy to see that the disaggregate model, with the inclusion of a single, spatial attribute, dramatically increases the dimensionality of the vector R_{th} .

The decision variable x_{th} consists of the vector of investments in energy conversion capacity x_{th}^{cap} and dispatch decisions x_{th}^{disp} . We assume we make decisions on energy investments only once a year at hour $h = 0$, which means we are only interested in x_{t0}^{cap} , with $x_{th}^{cap} = 0$

for $h > 0$. Energy dispatch decisions govern the flow of energy from each source node (e.g. a coal mine or source of natural gas), through conversion nodes (coal, nuclear and oil plants), through various transmission and storage nodes, and finally to demand nodes. A portion of the aggregate network is shown in figure 1. This is the network at a point in time (a particular hour). These networks are linked over time solely through the deposits to and withdrawals from storage devices. We let a be the attributes of a node in the dispatch network, which would represent moving energy over a link in the network. $x_{th,aa'}$ is the amount of energy moved from node a to node a' . We let $\theta_{aa'}$ capture transmission and storage losses when moving from a to a' ; θ can also handle changes in units (e.g. from barrels of oil to megawatthours of energy).

When we make decisions, they are governed by physical constraints. For example, there are conservation of flow constraints in the dispatch network, as well as upper bounds on the amount of energy that can be stored in a particular device or location at a point in time. The flow conservation constraints are given by

$$\sum_{a' \in \vec{\mathcal{A}}(a)} \theta_{a'a} x_{th,a'a}^{disp} - \sum_{a'' \in \vec{\mathcal{A}}(a)} x_{th,aa''}^{disp} = 0, \text{ for nonstorage locations,} \quad (2)$$

$$y_{tha} + \sum_{a' \in \vec{\mathcal{A}}(a)} \theta_{a'a} x_{th,a'a}^{disp} - \sum_{a'' \in \vec{\mathcal{A}}(a)} x_{th,aa''}^{disp} \geq 0. \text{ for storage locations.} \quad (3)$$

We let \mathcal{X}_{th} be the feasible region at time (t, h) , and we require that $x_{th} \in \mathcal{X}_{th}$.

Once we have made a decision, the system then evolves over time, with new information arriving that also changes the state of the system. We view new information as a random variable which represents some sort of exogenous change to one of our state variables. Thus, \hat{R}_{th} can be exogenous changes to our energy investments (Hurricane Katrina taking drilling rigs out of commission), \hat{D}_{th} can be random changes in demand for different types of electricity, \hat{p}_{th} can be the random change in the amount of rainfall, and $\hat{\rho}_{th}$ can be a change in technology, climate or policy (the government just imposed a carbon tax). These changes may take place on hourly time scales, but will often occur on longer time scales (we model technology as changing only once per year). We let W_{th} capture the vector of all these sources of random information that occurred in the hour preceding (t, h) .

Later, we need some language for describing a set of sample realizations. Let Ω be the set of all possible realizations of the sequence $(W_{11}, \dots, W_{1H}, W_{21}, \dots, W_{2H}, \dots, W_{TH})$ where T is the number of years and H is the number of time periods in a year. We let $W_{th}(\omega)$ be a sample realization of the random variables in W_{th} when we are following sample path ω . If we are following sample path ω^n for iteration n , we will index variables such as the state variable using S_{th}^n to represent the actual state at time (t, h) while following sample path ω^n .

All of the physics of the problem are captured by the transition function (also known as the system model, plant model or simply model) using

$$\begin{aligned} S_{t,h+1} &= S^M(S_{th}, x_{th}, W_{t,h+1}), \quad h = 0, 1, \dots, H-1, \\ S_{t+1,0} &= S^M(S_{tH}, x_{tH}, W_{t+1,0}). \end{aligned}$$

For example, changes in R_{th} and ρ_{th} would be written

$$R_{t,h+1} = R_{th} + x_{th}^{cap} + \hat{R}_{t,h+1}, \quad (4)$$

$$\rho_{t,h+1} = \rho_{th} + \hat{\rho}_{t,h+1}. \quad (5)$$

Our model assumes that capacity investment decisions made at the beginning of one year arrive at the beginning of the next year. In practice, an investment may take several years before it comes online. Even assuming a one-year delay requires that we make these decisions under uncertainty. If we wish to capture multi-year delays, we have an instance of a lagged asset acquisition problem. This issue has been addressed in an ADP setting in Godfrey and Powell (2002), Topaloglu and Powell (2006), and Nascimento and Powell (2009).

We note that it is possible to capture the dependence of exogenous information on the state of the system. For example, if we overinvest in ethanol capacity or wind turbines, we would expect that prices might drop. We did not explicitly do this in our numerical work, but the extension is a minor one. We may also have processes (e.g. prices, wind, demand) with autocorrelation. When this happens, we have to capture at least some of the history of these processes in the state variable. It is easy to incorporate these issues in the transition functions above, but it can complicate the challenge of designing good policies.

We measure our performance by adding up costs over time. Total capacity acquisition costs, dispatch costs, fuel costs and operating costs are given by

$$\begin{aligned} C_{th}^{cap}(S_{th}, x_{th}^{cap}) &= \text{The incremental capital costs in hour } h \text{ of year } t, \\ C_{th}^{disp}(S_{th}, x_{th}^{disp}) &= \text{The total costs resulting from the dispatch of energy in hour } h \\ &\quad \text{of year } t. \end{aligned}$$

We may express the total cost function in hour h of year t using,

$$C_{th}(S_{th}, x_{th}) = C_{th}^{cap}(S_{th}, x_{th}^{cap}) + C_{th}^{disp}(S_{th}, x_{th}^{disp}).$$

We have not addressed the problem of how to make decisions. In the next section, we present two models, a deterministic linear programming model, and a stochastic model, which lead to different methods for making a decision.

5. Optimization formulations

Our modeling framework above left unanswered the mechanism for actually making decisions. Section 5.1 describes a deterministic linear programming formulation that can be solved using commercial packages. Section 5.2 provides a stochastic optimization formulation, for which exact solution algorithms are not available. The remainder of the paper then focuses on designing an approximation strategy to solve the stochastic optimization model.

5.1. A deterministic linear programming model

If we assume that all the exogenous information is deterministic (and known in advance), we can formulate the decision problem as a single (albeit very large) linear program. The objective function is given by

$$\min_x \sum_{t \in \mathcal{T}} \sum_{h \in \mathcal{H}} C_{th}(S_{th}, x_{th}).$$

This has to be solved subject to two sets of constraints. The first set governs decisions made at a point in time, which is given by equations (14) - (19) in the online supplement, section

A.1. The second set of equations are the transition equations that link activities over time (basically the transition function), which are captured by equations (20) - (24) in the online supplement, section A.1. It is significant that the linear programming model views all these constraints together, over all points in time. This is why a linear programming model (even with deterministic formulations) quickly becomes too large to solve. For example, a twenty year, spatially aggregate model with hourly dispatch decisions produced a constraint matrix with over 34 million rows and 30 million columns.

5.2. A stochastic optimization model

The linear programming model suffers from two important limitations. First, it is unable to handle longer planning horizons while retaining the ability to model intermittent supplies and demand on an hourly level (along with storage). Second, it cannot handle the different forms of uncertainty that arise in energy policy analysis. For this reason, we pose the problem of controlling the stochastic version of the problem in terms of finding a policy (or decision function) to make decisions about dispatch, storage and investments over time. We represent the decision function using

$X_{th}^\pi(S_{th})$ = A function that returns a decision $x_{th} \in \mathcal{X}_{th}$ given the state S_{th} under the policy $\pi \in \Pi$.

The set Π refers to the set of potential decision functions, or policies, whose meaning becomes clearer below. At this point, we refer abstractly to the space of all possible decision functions.

The problem being stochastic, our goal is to find the optimal policy from among the set Π of policies by solving

$$\min_{\pi \in \Pi} \mathbb{E} \left\{ \sum_{t \in \mathcal{T}} \sum_{h \in \mathcal{H}} C_{th}(S_{th}, X_{th}^\pi(S_{th})) \right\}, \quad (6)$$

where $S_{t,h+1} = S^M(S_{th}, x_{th}, W_{t,h+1})$ (for $h = 0, \dots, H-1$) and $S_{t+1,0} = S^M(S_{tH}, x_{tH}, W_{t+1,0})$. The challenge, of course, is designing the policy $X_{th}^\pi(S_{th})$.

We note that by construction of our decision function (in particular, its dependence on the state variable) it is not allowed to depend on events in the future. This seemingly essential property is violated by the stochastic models that are allowed to make a decision as a function of a ‘‘scenario’’ which might specify, for example, the entire sample path of prices or energy from wind.

6. Algorithmic strategy

There are a number of strategies that have been used to solve stochastic optimization problems as stated in equation (6). These include various types of myopic policies (such as the logit investment rule in SEDS), rolling horizon policies (which use a point forecast of the future to make decisions now), simulation-optimization where we optimize the parameters of a myopic policy Swisher et al. (2000), stochastic programming (Higle and Sen (1996), Birge and Louveaux (1997), Wallace and Fleten (2003)), and dynamic programming. We

adopt the framework of dynamic programming in section 6.1, which introduces a number of computational hurdles. These are addressed in section 8 using approximate dynamic programming.

6.1. Dynamic Programming Formulation

Our strategy to solve the energy model involves formulating the model as a dynamic program, solving decision problems for each hour $h \in \mathcal{H}$ in each year $t \in \mathcal{T}$. We first define the value $V_{th}(S_{th})$ of a state S_{th} as the sum of the contributions that we expect to make, starting in state S_{th} , if we act optimally (make optimal decisions) until the end of the time horizon. Bellman’s equation enables us to recursively compute the optimal value functions associated with each state,

$$V_{th}(S_{th}) = \max_{x_{th} \in \mathcal{X}_{th}^{disp,n}} \left(-C_{th}(S_{th}, x_{th}) + \mathbb{E}(V_{t,h+1}(S_{t,h+1}) | S_{th}) \right), \quad (7)$$

where $S_{t,h+1} = S^M(S_{th}, x_{th}, W_{t,h+1})$.

As pointed out earlier, for a given year t , we assume that capacity additions are made only at the beginning ($h = 0$) of the year with new capacity arriving in some future year; the capacity resource state remains unchanged during the remaining hours. The value function $V_{th}(S_{th})$ in equation (7) captures the information about the future state of the system while solving the problem for the current time, including the value of water left behind in the reservoir after the decisions in the current time period are implemented.

In the remainder of the paper we consider only a single form of storage, namely the reservoir that stores water collected from precipitation. This means that the set \mathcal{A}^{Stor} consists of a single aggregate reservoir, but it is straightforward to adapt our model to multiple reservoirs, as well as multiple types of storage. The water from the reservoir may be used to satisfy the energy demands now or stored to be used to satisfy demands later in time. Hydro-energy is typically among the cheaper types of energy and tends to be used before more expensive energy types (such as coal or nuclear) if we do not provide for storage. Ideally, our model of reservoir management should mimic real-life behavior where water would be allowed to collect and be stored in the reservoir during periods of heavy precipitation to be used later during dry periods.

Solving the dynamic program in equation (7) is generally hard even for problems with small state spaces. The traditional dynamic programming approach for solving equation (7) suffers from three curses of dimensionality arising from the state space ($\{S_{th}\}_{h \in \mathcal{H}, t \in \mathcal{T}}$), the exogenous information ($\{W_{th}\}_{h \in \mathcal{H}, t \in \mathcal{T}}$), and the action space ($\{\mathcal{X}_{th}^{disp}\}_{h \in \mathcal{H}, t \in \mathcal{T}}$) (see Powell (2007), chapter 5). We now describe the techniques we use to overcome these.

We first define the post-decision state variable, $S_{th}^x = S^{M,x}(S_{th}, x_{th})$. The post-decision state for hour h is the state of the system immediately after making the decisions at hour h , but before the occurrence of the random events ($W_{t,h+1}$) between hours h and $h + 1$. In our analysis, we focus on two important components of the state variable, namely the capacity resource state R_{th} and the reservoir level y_{th} . The post-decision transitions of these

components can be written as,

$$\begin{aligned} R_{tha}^x &= R_{tha} + x_{tha}^{cap}, \quad a \in \mathcal{A}^{conv}, \\ y_{tha}^x &= y_{tha} + \sum_{a' \in \overleftarrow{\mathcal{A}}(a)} \theta_{a'a} x_{th,a'a}^{disp}, \quad a \in \mathcal{A}^{Stor}. \end{aligned}$$

We then construct the value function around the post-decision state $S_{th}^x = (R_{th}^x, y_{th}^x)$ using

$$V_{th}^x(S_{th}^x) = \mathbb{E}[V_{t,h+1}(S_{t,h+1}) | S_{th}^x]. \quad (8)$$

This allows us to avoid having to compute an expectation within the optimization formulation of equation (7). The modified dynamic program can be written as

$$V_{th}(S_{th}) = \max_{x_{th} \in \mathcal{X}_{th}^{disp}} \left(-C(S_{th}, x_{th}) + V_{th}^x(S_{th}^x) \right).$$

This leaves the problem of computing $V_{th}^x(S_{th}^x)$. For this, we turn to the techniques of approximate dynamic programming. It is important to realize that our strategy represents a significant departure from classical modeling and algorithmic strategies used in the approximate dynamic programming literature (notably Bertsekas and Tsitsiklis (1996) and Sutton and Barto (1998)) which approximate the value function around the pre-decision state. This strategy still leaves us with the challenge of computing or approximating the expectation. We avoid this by directly approximating the value function around the post-decision state.

6.2. An Approximate Dynamic Programming Algorithm

Our solution strategy involves choosing an appropriate functional form for the value functions. Our idea is very similar to that used in Pereira and Pinto (1991) for water reservoir management, with the significant difference that we are using separable, piecewise linear approximations of the value of resources in the future, where resources can be the amount of energy in storage as well as energy investments. This contrasts with the use in Pereira and Pinto (1991) of multidimensional cuts based on Benders decomposition, which carries with it provable convergence properties (examples of convergence proofs are given in Higle and Sen (1991) and Ruszczyński (1993)). We have compared both of these strategies and found that the separable approximations produced much faster convergence than approximations based on Benders cuts (see Powell et al. (2004) for an analysis in the context of two-stage problems) and very accurate solutions for complex multistage problems that arise in transportation (see Powell and Topaloglu (2003), Topaloglu and Powell (2006)). However, both strategies use the same basic idea of forming piecewise linear approximations based on dual information.

Starting with initial value function approximations, we step forward in time, solving the decision problem at each time using the value function approximation of the future states. We repeat this procedure over several iterations, updating the value function approximations after each iteration. This iterative updating enables us to form an approximation of the expectation in equation (8) (actually, we only need to approximate the slopes). While this algorithmic strategy is not generally optimal, there are convergence proofs for special

versions of the problem (discussed below), and in section 7 we benchmark against the linear programming formulation for a deterministic version of the problem which demonstrates very good agreement.

Using some policy π , we may write the objective function for time t as follows,

$$\begin{aligned} F_{th}^\pi(S_{th}) &= -C_{th}(S_{th}, X_{th}^\pi) - \sum_{h'>h} C_{th'}(S_{th'}, X_{th'}^\pi) - \sum_{t'>t} \sum_{h \in \mathcal{H}} C_{t'h}(S_{t'h}, X_{t'h}^\pi) \\ &= -C_{th}(S_{th}, X_{th}^\pi) + V_{th}^x(S_{th}^x), \end{aligned} \quad (9)$$

where $V_{th}^x(S_{th}^x) = -\sum_{h'>h} C_{th'}(S_{th'}, X_{th'}^\pi) - \sum_{t'>t} \sum_{h \in \mathcal{H}} C_{t'h}(S_{t'h}, X_{t'h}^\pi)$ is the same as the post-decision value function in equation (8). Since this function is unknown, we use an approximation which we denote as $\bar{V}_{th}^x(S_{th}^x)$. We now introduce the approximation that the value function is separable in the post-decision capacity resource state R_{th}^x and reservoir level y_{th}^x . We express this approximation as the sum of components denoting the value of the capacity resource state and the value of water remaining in the reservoir,

$$\begin{aligned} \bar{V}_{th}^x(S_{th}^x) &= \bar{V}_{th}^{cap}(R_{th}^x) + \bar{V}_{th}^{hydro}(y_{th}^x) \\ &= \sum_{a \in \mathcal{A}^{Conv}} \bar{V}_{tha}^{cap}(R_{tha}^x) + \bar{V}_{th}^{hydro}(y_{th}^x). \end{aligned}$$

It is easy to show that $V(S)$ is concave in R and y , and we approximate $V(R)$ and $V(y)$ using separable, piecewise linear functions (using a very fine discretization, since the quantities are continuous). It is important to recognize that we are only concerned about the derivative of $\bar{V}_{th}^x(S_{th}^x)$ rather than the actual value.

The hourly dispatch problem involves solving the optimization problem

$$x_{th}^{disp,n} = \arg \max_{x_{th} \in \mathcal{X}_{th}^{disp,n}} \left(-C_{th}^{disp}(S_{th}^n, x_{th}) + \bar{V}_{th}^{hydro}(y_{th}^x) \right) \quad (10)$$

where S_{th}^n is the specific state that we are in at hour h , year t , while following sample path ω^n . y_{th}^x is a function of both S_{th}^n and x_{th}^{disp} . Equation (10) defines our policy π , and the space of policies Π is the set of all possible value function approximations. The dispatch problem at time t is illustrated in figure 2, where the value function approximation (the value of water stored in the reservoir) is given by a piecewise linear function. It is important to emphasize that this is a very small linear program (for a spatially aggregate model), which scales fairly easily to spatially disaggregate applications (see Powell et al. (2002), Powell and Topaloglu (2005), Topaloglu and Powell (2006), Simao et al. (2009) for numerous transportation applications). This is important, since we have to solve 8,760 of these for each year of planning.

The annual capacity acquisition problem, solved at the beginning of each year, is given by

$$x_{t0}^{cap,n} = \arg \max_{x_{t0}^{cap}} \left(-C_{t0}^{cap}(S_{t0}^n, x_{t0}) + \bar{V}_{t0}^{cap}(R_{t0}^x) \right). \quad (11)$$

subject to

$$R_{t0}^x = R_{t0} + x_{t0}^{cap,n}, \quad (12)$$

$$x_{t0}^{cap,n} \geq 0. \quad (13)$$

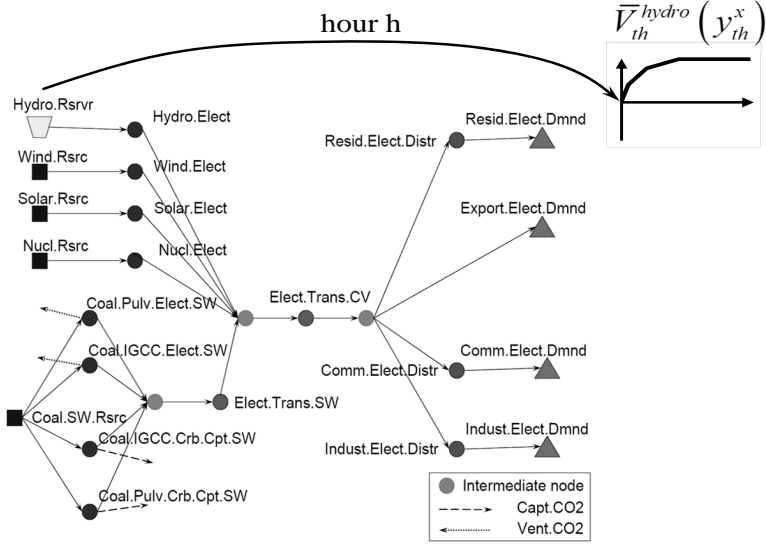


Figure 2: The hourly optimization problem with value function approximations for the reservoir level.

This problem is illustrated in figure 3. Here, we use a value function approximation for each type of energy resource. The only decision of how much to purchase is handled by the piecewise linear value function giving the value of resources in the future.

Using ADP, we learn the value function approximations using iterative updating operations. Let n denote the iteration counter while following sample path ω^n . For resource allocation problems, we have found that it is effective to compute the marginal value of an additional unit of a resource (this can be energy resources R_{th} , or the energy in storage y_{th}). Let

- $\hat{v}_{tha}^{cap,n}$ = A sample estimate of the marginal value of increasing R_{tha} by one unit computed while following sample path ω^n ,
- $\hat{v}_{tha}^{hydro,n}$ = A sample estimate of the marginal value of increasing y_{tha} by one unit computed while following sample path ω^n .

$\hat{v}_{tha}^{hydro,n}$ is computed relatively simply by finding the marginal impact of the value of additional energy in storage for a particular hour. Computing $\hat{v}_{tha}^{cap,n}$ requires calculating the incremental value of additional energy investment over an entire year. This requires finding the incremental value of, say, additional wind turbine generating capacity for each hour over the year, taking into account the presence of storage. This calculation allows us to capture the impact of hourly, daily and seasonal variability, as well as the marginal impact of additional storage.

In SMART, we used the updating equations given in Powell (2007), section 11.3. Note that the value $\hat{v}_{th}^{hydro,n}$ or $\hat{v}_{th}^{cap,n}$ found at time t , iteration n , is used to update the value function approximation around the post-decision state variable in the *previous* time period (see Powell (2007), section 4.4). We represent general updating operations for the value

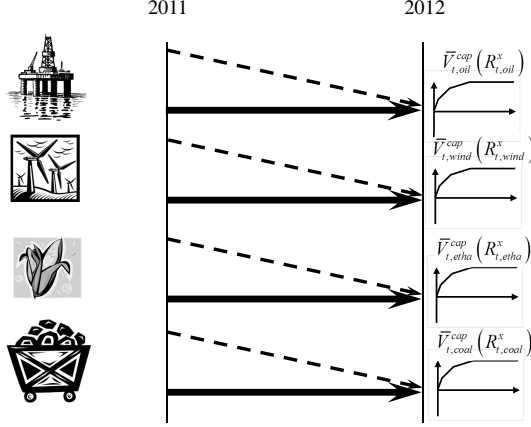


Figure 3: The annual capacity acquisition problem with value function approximations for the future capacity resource state.

functions as follows,

$$\begin{aligned} \bar{V}_{t,h-1}^{hydro,n}(y_{t,h-1}^x) &\leftarrow U^V \left(\bar{V}_{t,h-1}^{hydro,n-1}, y_{t,h-1}^{x,n}, \hat{v}_{th}^{hydro,n} \right), \\ \bar{V}_{t,h-1}^{cap,n}(R_{t,h-1}^x) &\leftarrow U^V \left(\bar{V}_{t,h-1}^{cap,n-1}, R_{t,h-1}^{x,n}, \hat{v}_{th}^{cap,n} \right). \end{aligned}$$

The details of the calculation of the derivatives $\hat{v}_{th}^{hydro,n}$ and $\hat{v}_{th}^{cap,n}$ are somewhat involved and are given in the online supplement, sections A.2 and A.3, respectively.

A complete summary of the approximate dynamic programming algorithm is given in figure 7, which was moved to the online appendix due to space constraints.

6.3. Discussion

The essential feature of the algorithm is that it steps forward through time, using a Monte Carlo sample of any random variable, solving sequences of very small linear programs. In this sense, the model scales well to handle more complex information processes, or a spatial representation of the problem (for example, dividing the country into 10 to perhaps 100 regions). If the model is being used over a 30 year horizon, in hourly increments, we would have to model over 250,000 time periods. This is quite manageable for an aggregate model (as described in our experimental work), but CPU times would start to become an issue with a sufficiently fine-grained spatial model.

Simulating a policy forward in time makes it very easy to handle considerable complexity in the transition function. We can handle complexities such as exogenous information such as prices which depend on decisions about the supply of energy generating capacity. We can also handle autocorrelations in wind, price and demand processes. However, adding dimensions to the state variable can potentially complicate the process of making decisions. Ideally, we would like to calculate a value function which has all the dimensions of the state variable, but at the same time we have to recognize that these additional dimensions may have little or no affect on the quality of a decision. These issues represent important areas of research that are beyond the scope of this paper.

We have illustrated the logic using value function approximations which ignore other aspects of the state variable such as the state of climate (is global warming progressing faster than expected), technology (was there a significant change in the performance of batteries or the cost of extracting and sequestering carbon), markets (is the price of oil over \$150) and government policy (is there a carbon tax).

We can handle these dimensions, in theory, by simply replacing the separable piecewise linear value functions $\bar{V}_{th}^{cap}(R_{th}^x)$ and $\bar{V}_{th}^{hydro}(y_{th}^x)$ with functions that also depend on the state variable ρ_{th} , as in $\bar{V}_{th}^{cap}(R_{th}^x|\rho_{th})$ and $\bar{V}_{th}^{hydro}(y_{th}^x|\rho_{th})$. The algorithm would be virtually unchanged (see the storage problem described in Nascimento and Powell (2009) for an illustration). The vector ρ_{th} evolves exogenously over time, and is already being computed. All that is needed is that we have to index the value functions by this information.

Of course, this is much easier to write than to implement. The number of possible values of ρ_{th} is extremely large, and as a result finding a value function approximation $\bar{V}_{th}^{cap}(R_{th}^x|\rho_{th})$ for each ρ_{th} is computationally impossible when ρ_{th} has more than two or three dimensions. There is a wide range of statistical strategies for handling this problem (see, for example, Hastie et al. (2009)) which represent promising avenues for research. There is also a growing base of literature in approximate dynamic programming that draws on techniques such as kernel regression (see Ormoneit and Sen (2002) and Fan and Gijbels (1996)) which offer considerable promise.

6.4. Convergence analysis

Our solution strategy is not optimal, but we offer both theoretical arguments and, in section 7, empirical evidence to support the claim that the quality of the solution is extremely good.

First, the algorithm for optimizing storage over a single year is an instance of a provably convergent algorithm, as given in Nascimento and Powell (2011) (building on the same proof technique as that given in Nascimento and Powell (2009)). This proof technique applies even if we index the value functions on the exogenous state vector ρ_{th} . Central to this convergence proof is that the storage (which is the only means by which one subproblem interacts with another) is a scalar (the amount stored in a reservoir). Just the same, optimizing the storage in water reservoirs is a particularly difficult problem, since we need to decide whether to store additional water in the rainy months of February and March to satisfy needs later in the summer, thousands of hours into the future.

Second, the use of separable, piecewise linear approximations has been proven to be optimal for two-stage problems if the second stage is separable, and near-optimal even when the second stage is nonseparable (see Powell et al. (2004)). Separable approximations are not, in general, optimal for nonseparable, multistage resource allocation problems. However, we can obtain provably convergent algorithms for multistage problems. If we were to introduce other forms of storage, the algorithm would only be an approximation due to our use of separable functions. But for practical uses, separable approximations have been found to scale very well to high dimensional problems with fast convergence.

Theoretical convergence proofs, while reassuring, are no guarantee of empirical performance. Our research has been preceded by extensive experimental research with algorithms of this type for transportation applications (see Powell and Godfrey (2002), Topaloglu and Powell (2006)). In the next section, we report on experimental comparisons of the ap-

proximate dynamic programming algorithm on deterministic and stochastic versions of the problem.

7. Experiments

In this section, we design experiments that focus on the ability of the ADP algorithm in section 8 to learn near-optimal policies in terms of how much to store, and what investment decisions to make. We begin by comparing ADP to LP for a deterministic version of the problem using an aggregate national model, which is small enough that we can solve the linear program. We also demonstrate that the algorithm can be run on a spatially disaggregate model, and show that CPU times scale sublinearly as the size of the network grows. We then demonstrate the algorithm on a stochastic problem. Our problem, with over 175,000 time periods, is far beyond the capabilities of any known algorithm for stochastic optimization, and as a result we do not have an optimal benchmark for this problem. But we do show that SMART produces robust behaviors.

We begin, in section 7.1, by describing the performance indices that we use to evaluate our algorithmic strategy. Section 7.2 reports on CPU times as we increase the number of time periods, and as we make the transition from a spatially aggregate to a disaggregate network. In section 7.3, we describe our experiments in a purely deterministic setting where we run tests over a single year planning horizon with deterministic precipitation. In section 7.4, we discuss experiments with stochastic precipitation scenarios. In section 7.5, we focus on experiments on capacity acquisition in a multi-decade setting.

7.1. Performance measures

In this section, we design performance measures to determine how closely the ADP algorithm matches the optimal solutions generated by an LP solver. We use three types of policies to construct our performance measures: ADP (the solution produced using approximate dynamic programming), OPT (the optimal solution of the deterministic problem produced by solving the linear program), and MYOPIC, which is the results of a myopic policy (the same as ADP with value function approximations set to zero). The myopic policy is used only to compute one of the performance metrics. Below, we let strategy s refer to ADP, OPT or MYOPIC.

We define the following terms that will be used for computing the statistics that we measure:

$$\begin{aligned}
 F^s &= \text{The objective function value as computed using strategy } s. \text{ For example, } \\
 &F^{MYOPIC} \text{ denotes the myopic solution and } F^{OPT} \text{ represents the optimal ob-} \\
 &\text{jective function value.} \\
 x_{tha}^s &= \text{The total usage corresponding to resource node with attribute vector } a \in \mathcal{A}^{Res} \\
 &\text{(} a^{Type} \in \{\text{coal, solar, nuclear, } \dots\} \text{) in hour } h \text{ of year } t \text{ according to strategy} \\
 &s, \\
 &= \sum_{a' \in \vec{\mathcal{A}}(a)} x_{th,aa'}^s.
 \end{aligned}$$

- y_{th}^s = The reservoir level in hour h of year t obtained by following strategy s .
- c_a = Cost of adding a unit of capacity of type a .
- x_{ta} = Amount of capacity type a added in year t using ADP.
- x_{ta}^* = The optimal amount of capacity type a to be added in year t .
- z_a^* = The optimal amount of capacity type a that is used in the starting year.

We use these to compute the following measures:
 General performance indices:

$$\begin{aligned}
 E_1^s &= \text{The error in the objective function (using strategy } s \text{) from the} \\
 &\quad \text{optimal objective function as a percentage of the optimal.} \\
 &= \frac{F^s - F^{OPT}}{F^{OPT}} \times 100\% \\
 E_2^s &= \text{The error in the objective function (using strategy } s \text{) as a per-} \\
 &\quad \text{centage of the error in the objective function for a myopic policy.} \\
 &= \frac{E_1^s}{E_1^{MYOPIC}} \times 100\%
 \end{aligned}$$

Performance indices for hydro-storage:

$$\begin{aligned}
 R_1^s &= \frac{\sum_{t,h,a} |x_{tha}^s - x_{thi}^{OPT}|}{\sum_{t,h,a} |x_{tha}^{OPT}|} \\
 R_2^s &= \frac{\sum_{t,h} (y_{th}^s - y_{th}^{OPT})^2}{\sum_{t,h} (y_{th}^{MYOPIC} - y_{th}^{OPT})^2}
 \end{aligned}$$

Indices for capacity-matching:

$$\begin{aligned}
 R_1^{cap} &= \frac{\sum_a c_a \sum_t |x_{ta} - x_{ta}^*|}{T \sum_a c_a z_a^*} \times 100\% \\
 R_2^{cap} &= \frac{\sum_a c_a |\sum_t x_{ta} - \sum_t x_{ta}^*|}{T \sum_a c_a z_a^*} \times 100\%
 \end{aligned}$$

E_2^s is a good indicator of the improvement in the solution quality obtained by using strategy s compared to a myopic solution. R_1^s measures the deviation of the solution from the optimal as a fraction of the optimal solution. R_2^s is a measure of how well strategy s improves over a myopic solution of the storage problem.

7.2. Execution times and scalability

A major feature of approximate dynamic programming is that it can scale to handle important dimensions such as fine-grained temporal and spatial modeling, as well as uncertainty. In this section, we summarize CPU times as we make the transition to handle problems with over 175,000 time periods, and as we move from a spatially aggregate to a disaggregate model. All runs were performed on a 2.5 Ghz, Intel Xeon processor with access to 64

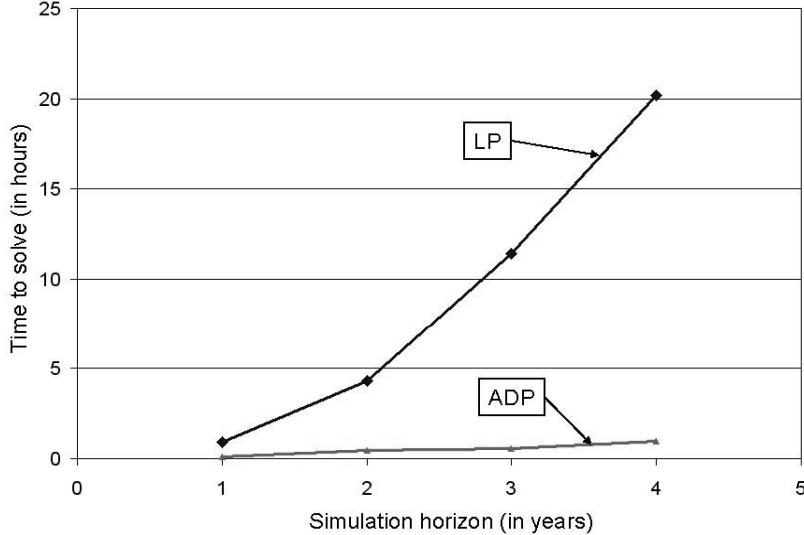


Figure 4: Time to solve the energy model as a function of the number of years in the simulation model. With increasing number of years in the model, run times increase dramatically in order to obtain an optimal solution while ADP produces near-optimal results in a fraction of that time.

gigabytes of memory. The linear program was solved using Cplex Version 12. The model is coded in Java.

We first ran a series of tests to demonstrate the ability of the ADP algorithm to handle a large number of time periods, which we compare to solving a single, large deterministic model. We emphasize that the energy modeling community feels that it is very important to model wind at hourly increments, since longer increments have the effect of averaging out the peaks and valleys that plague this important energy source. Figure 4 shows the computational time in hours to solve the linear program as a function of the number of years. With a horizon of four years (35,000 time periods), the linear program using the aggregate network required 20 hours, growing superlinearly with the horizon.

To compare ADP to the linear programming solution, we first performed a series of comparisons of the ADP objective function and solution (capacity investment and the amount held in storage). We found, using the aggregate model and a deterministic dataset, that we could always obtain solutions within one percent (and often within a fraction of a percent) using 300 iterations. Using this setting, figure 4 shows that ADP can solve the four-year horizon in under an hour.

We next turned to our spatially disaggregate model. This model represents energy generation and consumption at the level of the 48 states in the continental U.S., plus demand in the District of Columbia. It covers four types of energy supply (coal, nuclear, natural gas and wind) and four types of energy demand (residential, commercial, industrial and transportation). The network makes it possible to consider different types of technologies (e.g. investments in coal with carbon capture and sequestration). The resulting network has approximately 40,000 rows and columns.

Table 1 shows the size of the linear programs for the aggregate and disaggregate datasets. It then shows the CPU time (in seconds) for simulating a single year (8,760 subproblems),

	Aggregate dataset	Disaggregate dataset
Rows	198	40,229
Columns	174	39,705
CPU secs/first iteration	11.73	1,412.2
CPU secs/20 iterations	180.7	22,119.8
CPU secs/iteration	8.893	1,089.9
CPU secs/row/iteration	0.045	0.027

Table 1: Size of the linear programs for the aggregate and disaggregate networks (for a single time period), CPU times for first iteration over a year, for 20 iterations, average over last 19 iterations, and the CPU time per year normalized by the number of rows in the linear program.

for 20 iterations (of a year), the CPU seconds per iteration (as an average over the last 19 iterations), and the CPU seconds per iteration divided by the number of rows in the linear program. The last statistic provides an indication of how the CPU times grow with the size of the linear program. We note that the model builds the full linear program and solves it from scratch only once each year; for the remaining 8,759 subproblems, the only changes that occur are in the costs and right hand sides of the constraints, making it easy to use the previous solution.

The table shows that the CPU time per row, per year of simulation, actually drops as the problem size grows, from 0.045 CPU seconds/row (based on an average over the 20 year run) for the small aggregate dataset, to 0.027 CPU seconds/row for the large, disaggregate dataset.

Since ADP scales linearly with the number of iterations and number of time periods, we can quickly estimate run times as we increase the number of iterations. 300 iterations of the small network over a single year (which provided near optimal solutions when we compared the small network to the optimal solution) can be run in under an hour, making it possible to perform studies of wind penetration with different assumptions on storage. If the same run is extended over 20 years (to capture investment decisions), the run time grows to 15 hours, which is certainly manageable as an overnight run. 300 iterations of the large network over an entire year, while hopelessly intractable as a single deterministic model (350 million rows and columns) can be run in approximately 90 hours using ADP. This compares well to reports of run times of 30 days for some strategic planning models which solve daily unit commitment models over a year, but at this point it certainly seems to be an opportunity for parallel computation, an avenue we intend to pursue.

7.3. Experiments on hydro-storage

The first set of experiments focuses on testing how well the hydro-usage behavior of the solution from the ADP algorithm matches the optimal solution from the LP model using our aggregate dispatch model, under the assumption of deterministic precipitation. We focus on hydro storage because this is the most difficult form of storage from an algorithmic perspective. The reason for this complexity is that we may put water in storage in March so

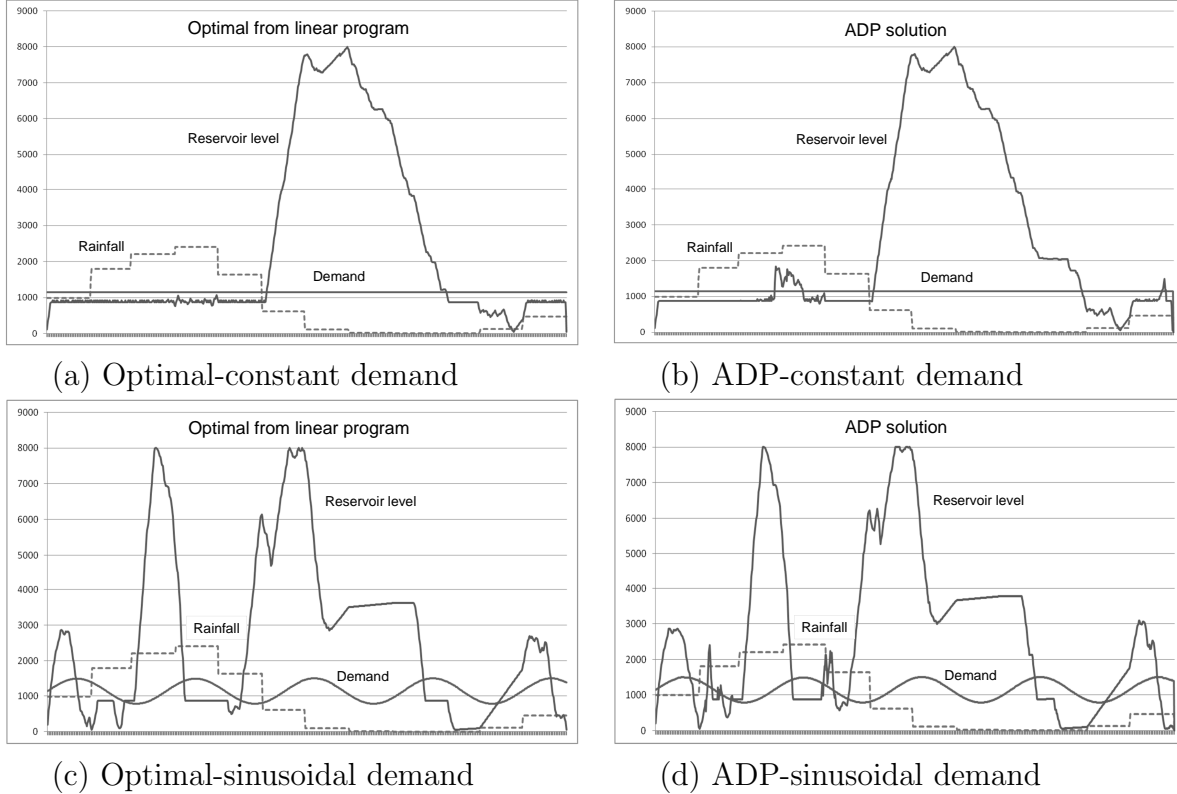


Figure 5: Optimal reservoir levels (a) and (c) and reservoir levels from ADP algorithm (b) and (d) for constant and sinusoidal demand.

that it can be used in August, thousands of hours in the future, which is a far more complex problem than storing energy in a battery at night so that it can be used during the day. We only solve the problem over a single year, since the reservoir empties out at least once each year.

We conducted our experiments in a variety of deterministic settings. The demands were (1) constant, (2) sinusoidally varying, or, (3) varying with no apparent pattern. Another setting that could be varied was whether or not wind (varying by the hour) was included as a competing source of energy. We tested our algorithm for a one-year horizon with storage value functions computed for every 10 hour increment. We then repeated the test for one hour increments. For some problems, this means that the decision of how much water to draw from the reservoir in March had to reflect usage behavior in July, thousands of hours into the future.

Figure 5 shows the reservoir level over the course of a one year period for a deterministic problem produced by solving the linear program which gives an optimal solution (figures (a) and (c)), and from the ADP algorithm (figures (b) and (d)). These were run for constant demand (figures (a) and (b)) and a sinusoidal demand (figures (c) and (d)). Note that the heaviest precipitation occurs during late winter and spring. The model used a slight penalty for holding water in the reservoir so that the optimal solution was to built up reserves as late as possible (this choice was made purely for algorithmic testing purposes). With this setup, the algorithm realized that it was necessary to begin building up reservoirs in the

March-April time frame to meet needs that would occur thousands of hours into the future. The results show an extremely close match between the optimal solution and that produced by the ADP algorithm.

In table 2, we summarize the performance statistics of the ADP algorithm. We observe that the ADP algorithm is able to produce solutions that are consistently within a very small margin of error of the optimal solutions.

Dataset	Demand type	H (hrs/timeperiod)	Wind	E_1^{ADP}	E_2^{ADP}	R_1^{ADP}	R_2^{ADP}
1	Constant	10	0	0.0008%	2.26%	0.02	0.006
2	Constant	10	✓	0.0003%	0.63%	0.04	0.007
3	Sinusoidal	10	0	0.0004%	0.02%	0.04	0.084
4	Sinusoidal	10	✓	0.0613%	2.67%	0.04	0.007
5	Fluctuating	10	0	0.0004%	0.04%	0.03	0.011
6	Fluctuating	10	✓	0.0000%	0.01%	0.03	0.005
7	Constant	1	0	0.0001%	0.06%	0.06	0.001
8	Constant	1	✓	0.0001%	0.08%	0.07	0.001
9	Sinusoidal	1	0	0.0012%	0.12%	0.03	0.002
10	Sinusoidal	1	✓	0.0020%	0.20%	0.03	0.001
11	Fluctuating	1	0	0.0000%	0.01%	0.02	0.001
12	Fluctuating	1	✓	0.0001%	0.02%	0.01	0.011

Table 2: Performance statistics of the ADP algorithm for different problem instances of a one-year energy model with the dispatch problem solved every H hours. 0 under wind means no wind, while check mark means wind was included.

7.4. Hydro-storage with stochastic precipitation

We now consider the case where the precipitation is stochastic. We generate 50 different precipitation scenarios for the year, and use these to train the value function approximations for approximate dynamic programming. In figure 6, we show the storage level produced by ADP for a single sample path, along with the storage level for each sample path when optimized knowing the entire sample path. The ADP solution has the intuitive behavior of building up storage earlier in the season to prepare for potentially lower rainfalls later in the season. By contrast, when we are allowed to optimize given the entire rainfall trajectory, the reservoir levels are held at lower levels until later in the season, reflecting the benefits of advance information. We believe that this illustrates the central weakness of models which sample uncertain information over the entire horizon, and then find the optimal solution given advance information.

7.5. Experiments on capacity acquisition

In the final set of experiments, we test the ability of the ADP algorithm to optimize capacity acquisition decisions over a multi-decade horizon under deterministic conditions. Given

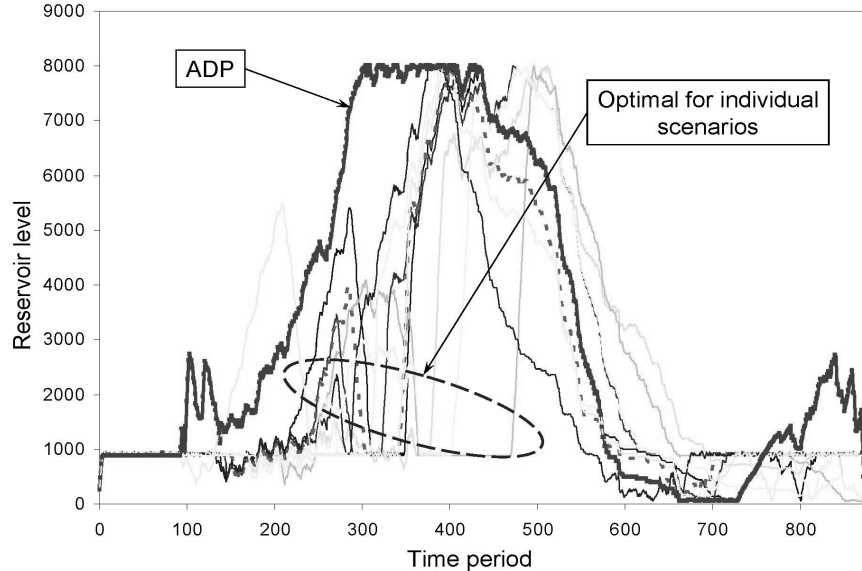


Figure 6: Reservoir level produced by ADP (last iteration) for stochastic precipitation, and optimal reservoir levels produced by LP when optimized over individual scenarios.

Dataset	Demand type	Wind	Years	E_1^{ADP}	E_2^{ADP}	R_1^{cap}	R_2^{cap}
1	Constant	0	5	0.03%	0.85%	0.008	0.008
2	Constant	0	20	0.52%	1.37%	0.031	0.067
3	Constant	✓	20	0.24%	0.72%	0.027	0.061
4	Fluctuating	0	20	0.34%	0.92%	0.029	0.061
5	Fluctuating	✓	20	0.30%	0.91%	0.026	0.056

Table 3: Performance statistics of the ADP algorithm for different problem instances of a multi-year energy model.

an initial set of capacities at the various conversion nodes, we allow the average demand to increase over the years. Eventually, the current level of capacities will be insufficient to cover the growing demands for energy. The ADP algorithm was found to produce a solution within 2.5 percent of the optimal solution after 100 iterations, 0.70 percent after 200 iterations, and 0.24 percent after 500 iterations.

In table 3, we summarize the performance statistics after 500 iterations of the ADP algorithm for a number of problem scenarios. Here again, the ADP algorithm is seen to approach within one percent of the optimal solution quality for a variety of deterministic datasets.

8. Conclusions

We have presented a stochastic, multiscale energy resource planning model that handles competing energy resources and demands, a wide range of forms of uncertainty and the ability to model hourly variations over a multidecade horizon. The computational research

shows that approximate dynamic programming can produce solutions that closely match the optimal solution of a deterministic linear programming model, while also providing solutions for problems that are far larger than what can be solved as a linear program.

The ADP model uses a proper model of the flow of information and decisions, and as a result avoids the short-cuts that have characterized many of the efforts at introducing uncertainty in optimization-based models. The experimental work shows that ADP produces robust behaviors such as storing rainfall earlier in the season than any of the solutions from a deterministic model acting on a specific scenario. It also exhibits nonanticipative behavior when faced with a major source of uncertainty such as the introduction of a carbon tax. While our algorithm is supported by some important theoretical results, we do not have a provably optimal algorithm for stochastic datasets which can serve as a comparable benchmark.

We show that the model can scale to handle spatially disaggregate problems, using a dataset which models each state in the continental U.S. We show that the model scales sublinearly with respect to the number of rows in the linear program for each time period, using the results of two networks with 200 and 40,000 rows.

The major unresolved algorithmic issue involves the handling of coarse-grained uncertainty such as the imposition of a carbon tax or a breakthrough in technology. ADP has no difficulty simulating different scenarios describing these sources of uncertainty, but estimating a value function approximation that takes these issues into account in a proper way represents a new area of research. However, using suitably designed approximation strategies, ADP can produce good solutions that are nonanticipative (do not peek into the future) and which properly capture the dynamics in a realistic way.

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