THRESHOLD RISK MEASURES PART 1: FINITE HORIZON

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ABSTRACT. In this paper we introduce the threshold risk measures, a class of risk measures incompatible with the coherent risk measures. In particular, the threshold risk measures consider the risk involved in applications where being above a threshold (for minimization problems) is considered too risky and should be avoided as much as possible. In this paper we develop the threshold risk measures together with their dynamic counterparts and apply these to the finite horizon dynamic programming with risk measures model. We develop several Bellman-type recursive algorithms to solve the finite horizon problem dynamic problem. A second part to this paper focuses on infinite horizon problems [10].

1. INTRODUCTION

The traditional approach of optimizing the expectation in stochastic and dynamic programs successfully introduces uncertainty of events in dynamic models but, in general, fails to convey the element of risk that many practical problems face. During the past 14 years researchers have developed the coherent risk measures as an alternative to the expectation operator in traditional stochastic programs. Coherent risk measures are consistent with the theory of risk developed for capital markets in the seminal work by Artzner, Delbaen, Eber, and Heath [3,4] and have a rich axiomatic theory allowing the development of efficient methods for the solution of risk-averse programs (see for example [2,9,20]). In [11,29–32] we can find a comprehensive treatment of coherent risk measures and risk-averse optimization including the development of multi-stage risk-averse programs.

Recently, increased attention has been paid to dynamic measures of risk which allow for risk-averse evaluation of streams of future costs or rewards (e.g., [5, 8, 11, 14, 16, 22, 25, 27–32]). The progression from risk-neutral to risk-averse dynamic models via coherent risk measures follows a natural path where coherent risk measures allow the risk evaluation of uncertain outcomes. Combining these measures with risk-neutral dynamic programs gives rise to dynamic risk-averse optimization problems. Although natural, this progression is by no means trivial, requiring a full decade of development to reach the state where risk-averse dynamic models via coherent risk measures are postulated and applied, see [1, 2, 9, 20, 21, 32]. We could say that a high point of this research is attained at [27], where A. Ruszczyński develops risk-averse dynamic programs for Markov decision processes utilizing a dynamic version of coherent risk measures. In his paper, A. Ruszczyński gives solutions to finite and infinite horizon versions of such risk-averse dynamic programs via generalizations to the Bellman equation, value iteration and policy iteration algorithms. This work successfully establishes dynamic

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coherent risk measures as a risk-averse alternative to classical dynamic programming for applications where coherent risk measures apply. That is, applications where the notion of risk conveyed by coherent risk measures is meaningful, e.g. applications to risky capital markets and portfolio optimization.

In this paper we focus on a notion of risk incompatible with coherent risk measures and develop risk-averse dynamic models for it. In particular, we consider the risk involved in applications where being above a fixed threshold (for minimization problems) is considered too hazardous and should be avoided as much as possible. With this idea in mind we develop the threshold risk measures which, simply speaking, penalize random variables that take values above a given threshold.

To understand why the coherent risk measures are not suitable for applications involving thresholds we just have to consider how coherent risk measures behave in the face of a threshold $\alpha > 0$. Let Z be an arbitrary bounded random variable and let t > 0be such that $||tZ||_{\infty} < \alpha$. For a coherent risk measure ρ to represent the risk-averse attitude of focusing *purely* on staying below the threshold α , it is essential that $\rho(tZ) = 0$. Then by the properties of coherent risk measures we obtain that $0 = \rho(tZ) = t\rho(Z)$, and this implies that $\rho(Z) = 0$. In this way we can conclude that $\rho(Z') = 0$ for every bounded random variable Z'. This would imply that on finite probability spaces the only risk measure that we can apply is the 0-function!

Our definition of threshold risk measures bypasses this issue while still maintaining many of the desirable properties of coherent risk measures. We specifically retain related versions of all the properties that according to Artzner et al. [3,4] (and most of the risk measures community) make optimization problems based on coherent risk measures convey an idea of risk aversion. We accomplish this by extending the definition of coherent risk measures in a seemingly natural way. What we assumed would be a uncomplicated extension to the theory of coherent risk measures quickly grew into a theory of its own, due mostly to the numerous details that have to be handled when dealing with functions of random variables. The end result being a theory of threshold risk measures that parallels its coherent counterparts and simultaneously extends the theory of risk in new directions.

The notion of threshold risk came to our attention while working on applications related to electricity markets that face random spikes in the electricity spot prices. In these applications it is important for the decision maker to avoid decision policies where the cost outweighs the reward, i.e. the final cost is more than zero. In our applied work we implement the theory and methods from this paper combined with approximate dynamic programming techniques from [23] to the problem of hedging consumers against the volatility of purchasing electricity on the spot market.

Another interesting example arises in managing heating systems in commercial buildings in the presence of capacity pricing policies which penalize usage above a particular level. Some of our applied work implements models involving threshold risk measures to these problems on managing electricity expenditures in a network of high rise buildings in Manhattan. Both of these applications will appear in future papers where the details of the implementations and its numerical results will be given.

We would like to point out that we are not the only ones to propose these type of threshold risk measures as suitable measures of risk. Recently, [33] shows an approach similar to ours to develop risk-averse dynamic models for the hydrothermal system operation planning of the Brazilian interconnected power system. In this study, the authors use a specific threshold risk measure to design their risk-averse models and methods. By restricting themselves to the application at hand, the authors avoid many of the technical details needed to generalize the notion of threshold risk measures in a way that it is conducive to more applications.

Our main goal in this paper is to define the class of risk measures we refer to as *threshold risk measures* together with its dynamic counterparts, and bring the development of such to the state of the art presented by A. Ruszczyński in [27]. As a result we give a solid theoretical foundation to the use of threshold risk measures in dynamic applications such as the ones mentioned before. To accomplish this we leverage the decomposition method for dynamic coherent risk measures developed in [27] and extend it to the case of threshold risk measures. In this first part of our research we focus on finite horizon problems while a second paper extends the idea to infinite horizon problems [10].

The paper is organized as follows. In section 2 we carefully define the controlled Markov decision process on which we base our dynamic models. Section 3 introduces the spaces of random outcomes that form the basis of most of our definitions. Section 4 defines threshold risk measures and presents some of its main properties including two representation theorems that allow the development of dynamic models. In section 5 we define the dynamic version of threshold risk measures and prove its time consistency, a property essential to the development of dynamic programs that can be solved recursively without the need to backtrack decisions. Section 6 introduces the finite horizon risk-averse dynamic problem and a solution via a generalization of Bellman's equation. In this section we introduce Ruszczyński's decomposition method and prove that together with threshold risk measures it has the properties required for the development of the Bellman's equation. This section also contains a reformulation of the finite horizon problem in terms of the post decision state variable. In section 6.4 we explore the case where the risk in our stream of decisions is incurred only at the very last step. Finally, section 7 considers the case of static threshold risk measures and looks into choosing via optimization methods the most robust threshold sequence for our problems.

2. A CONTROLLED MARKOV DECISION PROCESS

In this section we define a controlled Markov decision process and introduce our notation based on the presentation of [17,27]. Let (S, \mathcal{B}_S) and $(\mathcal{A}, \mathcal{B}_A)$ be Borel spaces. We call S the state space and \mathcal{A} the action space. To each state $s \in S$ we associate a set of admissible actions $A(s) \subseteq \mathcal{A}$ in such a way that the map $s \mapsto A(s)$ defines a measurable multifunction. We call the multifunction $A(\cdot)$ an action set and define its graph as

$$graph(A) = \{(s, a) \in \mathcal{S} \times \mathcal{A} \mid a \in A(s)\}.$$

Let \mathcal{P} denote the set of probability measures on $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$ endowed with the usual weak topology. A *controlled kernel* is a measurable function $Q : \operatorname{graph}(A) \to \mathcal{P}$. So, for every state $s \in \mathcal{S}$ and action $a \in A(s)$ the value of Q(s, a) is a probability measure on the state space $(\mathcal{S}, \mathcal{B}_{\mathcal{A}})$. This can be interpreted as the probability of reaching a state given that we are in state s and take the action $a \in A(s)$, which for a Borel set $B \subseteq \mathcal{S}$ is denoted by $Q(B \mid s, a)$. A *cost function* is a measurable function $c : \operatorname{graph}(A) \to \mathbb{R}$. Our controlled Markov model has a state pace S, an action space A and sequences of action sets A_t , controlled kernels Q_t , and cost functions c_t , t = 0, 1, 2, ..., A Markov policy (or simply a policy) is a sequence of measurable functions $\pi_t : S \to A$, such that $\pi_t(s) \in A_t(s)$, for all $s \in S_t$ and all t = 0, 1, 2..., A Markov policy is stationary if $\pi_t = \pi_0$, for all t = 1, 2, ...

In this paper we assume that the initial state s_0 is fixed. This is not a big restriction and serves to ease our exposition.

Let Π be the set of all policies. Each policy results in a cost sequence that we could optimize. The classical *finite horizon expected value problem* looks for the policy $\pi^* = \{\pi_0^*, \ldots, \pi_{T-1}^*\}$ that minimizes the expected cost:

(2.1)
$$\min_{\pi \in \Pi} \mathbb{E}\left[\sum_{t=0}^{T-1} c_t \left(s_t, \pi_t(s_t)\right) + c_T(s_T)\right],$$

where $\pi = (\pi_0, \ldots, \pi_{T-1})$ and $c_T : S \to \mathbb{R}$ is a measurable function of *final cost*.

The infinite version of the problem above is known as the *infinite horizon discounted* expected value problem. For $\gamma \in (0, 1)$, it looks for a policy $\{\pi_t\}_{t=1}^{\infty}$ that minimizes the α -discounted expected value problem:

(2.2)
$$\min_{\pi \in \Pi} \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t c_t \left(s_t, \pi_t(s_t) \right) \right],$$

where similar to before, $\pi = {\pi_t}_{t=0}^{\infty}$.

Under some reasonable assumptions both of these problems have optimal Markov policies. The finite horizon problem (2.1) has an optimal policy that can be described by the famous Bellman's dynamic programming equations (see [24]). The infinite horizon problem (2.2) has a stationary optimal policy as long as its underlying Markov process is stationary too (see [24]). In this case the policy and value iteration algorithms converge to an optimal stationary policy.

Our goal is to add elements of risk-aversion to problems (2.1) and (2.2) by the use of a risk evaluation method based on threshold risk measures (to be defined in section 4). In this paper we focus on the finite horizon problem (2.1) and in [10] we tackle the infinite horizon problem. In the next few sections we develop the threshold risk measures and a nested risk evaluation method based on them.

3. Spaces of Random Outcomes

In this section we define the spaces of random outcomes that we use through the rest of the paper. Although presented in a general sense, it is useful to keep in mind that our purpose is to give a solid foundation to the stage-wise random costs inherent in dynamic optimization problems. The importance of properly establishing this is apparent once we realize that it is on these random outcomes that our risk measures must act.

Let (Ω, \mathcal{F}, P) be a probability space with σ -algebra \mathcal{F} and probability measure P. For every $p \in [1, +\infty)$, let $\mathcal{L}_p(\Omega, \mathcal{F}, P)$ be the set of P-measurable functions $f : \Omega \to \mathbb{R}$ whose p-th order moment is well defined, i.e.

(3.1)
$$||f||_p = \left(\int_{\Omega} |f|^p dP\right)^{1/p} < \infty.$$

We define $\mathcal{L}_{\infty}(\Omega, \mathcal{F}, P)$ as the set of essentially bounded *P*-measurable functions $f : \Omega \to \mathbb{R}$. For $f \in \mathcal{L}_{\infty}(\Omega, \mathcal{F}, P)$ define

(3.2)
$$||f||_{\infty} = \inf \left\{ c \ge 0 \mid |f(\omega)| \le c \text{ for almost all } \omega \in \Omega \right\},$$

then $\|\cdot\|_{\infty}$ is a norm on $\mathcal{L}_{\infty}(\Omega, \mathcal{F}, P)$.

Fix $p \in [1, +\infty)$. In this paper we restrict ourselves to the spaces of uncertain outcomes $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$ where each element $Z \in \mathcal{Z}$ is viewed as an uncertain outcome on (Ω, \mathcal{F}) and is by definition a random variable with finite *p*-th order moment with respect to the reference probability measure *P*.

For $Z, Z' \in \mathcal{Z}$ we denote by $Z \preceq Z'$ the pointwise partial order, meaning $Z(\omega) \leq Z'(\omega)$ for a.e. $\omega \in \Omega$. In this paper the variables Z represent random costs and so we prefer smaller realizations of it.

With each space $\mathcal{Z} := \mathcal{L}_p(\Omega, \mathcal{F}, P)$ there is an associated *conjugate dual* space $\mathcal{Z}^* := \mathcal{L}_q(\Omega, \mathcal{F}, P)$, where $q \in (1, +\infty]$ is such that 1/p + 1/q = 1. For $Z \in \mathcal{Z}$ and $\zeta \in \mathcal{Z}^*$ their scalar product is defined as

(3.3)
$$\langle \zeta, Z \rangle = \int_{\Omega} \zeta(\omega) Z(\omega) dP(\omega).$$

For $p \in [1, +\infty)$ we endow the space $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$ with either its strong (norm) or weak* topology. If $p = +\infty$, then we endow $\mathcal{Z} = \mathcal{L}_{\infty}(\Omega, \mathcal{F}, P)$ with its weak* topology. On the other hand, we always endow dual spaces $\mathcal{Z}^* = \mathcal{L}_q(\Omega, \mathcal{F}, P)$ with its weak* topology.

With these definitions \mathcal{Z} and \mathcal{Z}^* are *paired*, *locally convex* Banach spaces compatible with the scalar product (3.3). That is, every continuous linear functional on \mathcal{Z} can be represented as $\langle \zeta, \cdot \rangle$, for some $\zeta \in \mathbb{Z}^*$, and every continuous linear functional on \mathcal{Z}^* can be represented as $\langle \cdot, \mathbb{Z} \rangle$, for some $\mathbb{Z} \in \mathcal{Z}$, see [29].

4. Threshold Risk Measures

In this section we define a risk measure that penalizes random variables that take values above a given threshold function (for minimization problems). Before doing this we need to introduce a simple yet very important function on random variables.

4.1. Threshold Functions and the Nonnegative Operator. For $a \in \mathbb{R}$, let $[a]_+ = \max\{a, 0\}$. For a random variable $Z \in \mathcal{L}_p(\Omega, \mathcal{F}, P)$, define $[Z]_+$ as the random variable given by $[Z]_+(\omega) = [Z(\omega)]_+, \forall \omega \in \Omega$. We call $[\cdot]_+$ the *nonnegative* operator.

Definition 1. A threshold function is an \mathcal{F} -measurable nonnegative random variable $\alpha : \Omega \to \mathbb{R}_+$ such that $\alpha \in \mathcal{L}_p(\Omega, \mathcal{F}, P)$.

The properties of the nonnegative operator $[\cdot]_+$ are fundamental to our analysis. Here we list all the properties of $[\cdot]_+$ that we use through the paper and defer the proofs to the appendix.

Theorem 1. Let $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$ and let $\alpha \in \mathcal{Z}$ be a threshold function. Then for every $Z, W \in \mathcal{Z}$, the nonnegative operator $[\cdot]_+$ satisfies:

- (N0) Closeness: $[Z \alpha]_+ \in \mathcal{Z};$
- (N1) Convexity: $[tZ + (1-t)W]_+ \leq t[Z]_+ + (1-t)[W]_+$, for all $t \in [0,1]$;
- (N2) Monotonicity: If $Z \leq W$, then $[Z]_+ \leq [W]_+$;
- (N3) Positive homogeneity: If t > 0, then $[tZ]_+ = t[Z]_+$;

(N4) Subadditivity: $[Z + W]_+ \preceq [Z]_+ + [W]_+;$ (N5) If t > 0, then $[tZ - \alpha]_+ = t[Z - \alpha/t]_+;$

We have two important properties of $[\cdot]_+$ to introduce.

Theorem 2. Let $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $\alpha \in \mathcal{Z}$ be a threshold function, and let $\mathfrak{B} \in \mathcal{F}$ be a random event of the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $\mathbb{1}_{\mathfrak{B}}$ denote the indicator function of \mathfrak{B} . Then for every $Z \in \mathcal{Z}$,

$$[\mathbb{1}_{\mathfrak{B}}Z - \alpha]_{+} = \mathbb{1}_{\mathfrak{B}}[Z - \alpha]_{+}.$$

Theorem 3 (Linear representation of the nonnegative operator). Let $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$ and $\alpha \in \mathcal{Z}$ be a threshold function. Then for any $Z \in \mathcal{Z}$, the random variable $[Z - \alpha]_+$ can be obtained by solving the following optimization problem on random variables:

(4.1)
$$\begin{array}{c} \min_{X} X \\ s.t. \ X \succeq Z - \alpha \\ X \succeq 0, \ X \in \mathcal{Z} \end{array}$$

4.2. **Definition and Basic Properties of Threshold Risk Measures.** In this section we define and introduce the basic properties of threshold risk measures. Among others, we give two representation theorems that are fundamental in the efficient evaluation of the risk measures. The proofs of most of these theorems are deferred to the appendix.

A risk measure is a proper class function $\rho : \mathbb{Z} \to \overline{\mathbb{R}}$. By this we mean that ρ is constant on the classes of functions which differ only on sets of *P*-measure zero. The function ρ is proper in the sense that $\rho(Z) > -\infty$ for all $Z \in \mathbb{Z}$ and its domain

$$\operatorname{dom}(\rho) = \{ Z \in \mathcal{Z} : \rho(Z) < +\infty \}$$

is nonempty. A coherent risk measure is a risk measure $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ satisfying the following axioms:

- (A1) Convexity: $\rho(tZ + (1-t)W) \le t\rho(Z) + (1-t)\rho(W)$, for all $Z, W \in \mathbb{Z}$ and all $t \in [0, 1]$;
- (A2) Monotonicity: If $Z, W \in \mathcal{Z}$ and $Z \preceq W$, then $\rho(Z) \leq \rho(W)$;
- (A3) Translation equivariance: If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z + a) = \rho(Z) + a$;
- (A4) Positive homogeneity: If t > 0 and $Z \in \mathcal{Z}$, then $\rho(tZ) = t\rho(Z)$.

Coherent risk measures are the basic building blocks for our theory. A thorough exposition of it can be found in [32]

Definition 2. Let $p \in [1, +\infty)$, $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, and let $\varrho : \mathcal{Z} \to \mathbb{R}$ and $\vartheta : \mathcal{Z} \to \mathbb{R}$ be real-valued, lower semicontinuous coherent risk measures. Let $\eta > 0$ and $\alpha \in \mathcal{Z}$ be a threshold function. A risk measure of threshold α is defined by

(4.2)
$$\rho^{\alpha}(Z) = \varrho(Z) + \eta \,\vartheta([Z - \alpha]_+),$$

for every $Z \in \mathcal{Z}$. We call η the risk factor of ρ^{α} . For the sake of simplicity we also call this function a threshold risk measure (or TRM for short).

Let us show some examples of threshold risk measures.

Example 1. Let $\eta > 0$, $\mathcal{Z} = \mathcal{L}_1(\Omega, \mathcal{F}, P)$, and $\alpha \in \mathcal{Z}$ be a threshold function. The simplest threshold risk measure is the *mean upper semideviation from target* α given by

(4.3)
$$\rho^{\alpha}(Z) = \mathbb{E}(Z) + \eta \mathbb{E}([Z - \alpha]_{+})$$

for every $Z \in \mathcal{Z}$. Despite its simplicity, the mean upper semideviation from target appears organically in problems involving risk on electricity markets where the random spikes in price make a priority to optimize while staying above a natural predefined threshold function.

Example 2. Let $p \ge 1$, $\eta > 0$, $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, and $\alpha \in \mathcal{Z}$ be a threshold function. Define the *threshold exponential risk measure* by

(4.4)
$$\rho_e^{\alpha}(Z) = \mathbb{E}(Z) + \eta \inf_{\tau > 0} \tau \ln \mathbb{E}\left(e^{\tau^{-1}[Z-\alpha]_+}\right),$$

for every $Z \in \mathcal{Z}$. The threshold exponential risk measure provides an alternative to the mean upper semideviation from target with a "heavier" weight on the risk portion of the measure. See [32, ex. 6.17] for details.

Now we introduce the basic properties of threshold risk measures and two representation theorems. The proofs of these theorems are deferred to the appendix.

Theorem 4. Let $\rho^{\alpha} : \mathbb{Z} \to \mathbb{R}$ be a threshold risk measure with threshold function $\alpha \in \mathbb{Z}$. Then ρ^{α} is a real-valued, continuous and subdifferentiable risk measure on \mathbb{Z} . Moreover, ρ^{α} satisfies the following properties:

- (T1) Convexity: $\rho^{\alpha}(tZ + (1-t)W) \leq t\rho^{\alpha}(Z) + (1-t)\rho^{\alpha}(W)$, for all $Z, W \in \mathbb{Z}$ and all $t \in [0,1]$;
- (T2) Monotonicity: If $Z, W \in \mathcal{Z}$ and $Z \preceq W$, then $\rho^{\alpha}(Z) \leq \rho^{\alpha}(W)$;
- (T3) Translation equivariance: If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho^{\alpha}(Z + a) = \rho^{\alpha a}(Z) + a$;
- (T4) Positive homogeneity: If t > 0 and $Z \in \mathcal{Z}$, then $\rho^{\alpha}(tZ) = t\rho^{\alpha/t}(Z)$.

Notice that in (T3) we are not requiring the random variable $\alpha - a$ to be nonnegative, therefore $\alpha - a$ is a threshold function only when $\alpha - a \succeq 0$.

The TRM also hold some extra properties regarding its threshold functions.

Theorem 5. Let $\alpha, \beta \in \mathbb{Z}$ be threshold functions. Then the threshold risk measure satisfies the following properties:

- (C1) Convexity of threshold functions: $\rho^{\lambda\alpha+(1-\lambda)\beta}(Z) \leq \lambda\rho^{\alpha}(Z) + (1-\lambda)\rho^{\beta}(Z)$, for all $Z \in \mathcal{Z}$ and all $\lambda \in [0,1]$;
- (C2) Monotonicity of threshold functions: if $\alpha \leq \beta$ and $Z \in \mathcal{Z}$, then $\rho^{\alpha}(Z) \geq \rho^{\beta}(Z)$.

The following representation theorem for TRM relies on the Fenchel-Moreau Theorem (see [32, thm. 6.4, 7.7]) and the basic properties introduced in Theorem 4.

Theorem 6 (First Representation Theorem of TRM). Let (Ω, \mathcal{F}, P) be a sample space with sigma algebra \mathcal{F} and probability measure P. Let $p \in [1, +\infty)$, $q \in (1, +\infty]$ be such that 1/p + 1/q = 1 and let $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $\mathcal{Z}^* = \mathcal{L}_q(\Omega, \mathcal{F}, P)$ be a conjugate pair of spaces. Let $\alpha \in \mathcal{Z}$ be a threshold function and let $\rho^{\alpha} : \mathcal{Z} \to \mathbb{R}$ be a threshold risk measure with conjugate dual $(\rho^{\alpha})^* : \mathcal{Z}^* \to \mathbb{R}$. Then there exists a set \mathfrak{A} such that for every random variable $Z \in \mathcal{Z}$:

(4.5)
$$\rho^{\alpha}(Z) = \sup_{\zeta \in \mathfrak{A}} \left\{ \langle \zeta, Z \rangle - (\rho^{\alpha})^*(\zeta) \right\}, \quad \forall Z \in \mathcal{Z},$$

such that

$$\mathfrak{A} \subseteq \left\{ \zeta \in \mathcal{Z}^* \mid \zeta \succeq 0 \right\}.$$

The following theorem leverages the representation theorem of coherent risk measures [32, thm. 6.4] and the $[\cdot]_+$ operator to obtain a computationally tractable representation for TRM.

Theorem 7 (Second Representation Theorem of TRM). Let (Ω, \mathcal{F}, P) be a sample space with sigma algebra \mathcal{F} and probability measure P. Let $p \in [1, +\infty)$, $q \in (1, +\infty]$ be such that 1/p + 1/q = 1 and let $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $\mathcal{Z}^* = \mathcal{L}_q(\Omega, \mathcal{F}, P)$ be a conjugate pair of spaces. Let $\eta > 0$, $\alpha \in \mathcal{Z}$ be a threshold function, and let $\varrho : \mathcal{Z} \to \mathbb{R}$, and $\vartheta : \mathcal{Z} \to \mathbb{R}$ be real-valued, lower semicontinuous coherent risk measures. Then the threshold risk measure given by

$$\rho^{\alpha}(Z) = \varrho(Z) + \eta \,\vartheta([Z - \alpha]_+), \,\,\forall Z \in \mathcal{Z},$$

can be obtained by solving the following optimization problem:

(4.6)
$$\sup_{\mu,\zeta} \min_{X} \langle \mu, Z \rangle + \eta \langle \zeta, X \rangle$$
$$s.t. \ X \succeq Z - \alpha$$
$$X \succeq 0, X \in \mathcal{Z}$$
$$\mu \in \partial \varrho(0), \ \zeta \in \partial \vartheta(0),$$

where the subdifferentials $\partial \varrho(0)$ and $\partial \vartheta(0)$ are closed convex sets of probability density functions on \mathcal{Z} with respect to the reference probability P.

Due to the convexity of the subdifferentials $\partial \varrho(0)$ and $\partial \vartheta(0)$, representation (4.6) is a convex optimization problem with linear objective function to which we can apply specialized techniques to efficiently obtain a solution.

Below we show the representations given by theorems 6 and 7 applied to the examples of threshold risk measures presented before.

Example 3. Let $\eta > 0$, $\mathcal{Z} = \mathcal{L}_1(\Omega, \mathcal{F}, P)$, and $\alpha \in \mathcal{Z}$ be a threshold function. The conjugate dual to \mathcal{Z} is the space $\mathcal{Z}^* = \mathcal{L}_{\infty}(\Omega, \mathcal{F}, P)$ of essentially bounded *P*-measurable random variables. Consider the mean upper semideviation from target α given by

(4.7)
$$\rho^{\alpha}(Z) = \mathbb{E}(Z) + \eta \mathbb{E}([Z - \alpha]_{+}),$$

for every $Z \in \mathcal{Z}$.

Suppose now that the threshold function α is constant. Then

$$\mathbb{E}\left([Z-\alpha]_{+}\right) = \sup_{\substack{\|\zeta\|_{\infty} \leq 1 \\ \|\zeta\|_{\infty} \leq 1 \\ \zeta(\cdot) \geq 0}} \mathbb{E}\left(\zeta[Z-\alpha]_{+}\right)$$

therefore

$$\rho^{\alpha}(Z) = \mathbb{E} (Z) + \eta \mathbb{E} ([Z - \alpha]_{+}) \\
= \mathbb{E} (Z) + \sup_{\substack{\|\zeta\|_{\infty} \leq \eta \\ \zeta(\cdot) \geq 0}} \mathbb{E} (\zeta Z - \alpha \zeta) \\
= \sup_{\substack{\|\zeta\|_{\infty} \leq \eta \\ \zeta(\cdot) \geq 0}} \{\mathbb{E} (Z) + \mathbb{E} (\zeta Z) - \alpha \mathbb{E} (\zeta)\} \\
= \sup_{\substack{\|\zeta\|_{\infty} \leq \eta \\ \zeta(\cdot) \geq 0}} \{\langle \mathbb{1} + \zeta, Z \rangle - \alpha \mathbb{E} (\zeta)\} \\
= \sup_{\substack{\zeta \in \mathfrak{A}}} \{\langle \zeta, Z \rangle - \alpha \mathbb{E} (\zeta - \mathbb{1})\},$$

where

(4.9)
$$\mathfrak{A} = \{\zeta + \mathbb{1} \mid \zeta \in \mathcal{Z}^*, \, \|\zeta\|_{\infty} \le \eta, \, \zeta \succeq 0\}.$$

In this way we see that first representation (4.5) holds with \mathfrak{A} given by (4.9) and $(\rho^{\alpha})^*(\zeta) = \alpha \mathbb{E}[\zeta - 1], \forall \zeta \in \mathcal{Z}^*.$

For $\varrho(Z) = \mathbb{E}(Z)$ and $\vartheta(Z) = \mathbb{E}(Z)$ it is not difficult to see that the sets $\partial \varrho(0)$ and $\partial \vartheta(0)$ are singletons consisting of the density with respect to the reference probability P, i.e. the Radon-Nikodym derivative with respect to P. Denoting these densities by μ and ζ , respectively, we get that $\varrho(Z) = \langle \mu, Z \rangle$ and $\vartheta(Z) = \langle \zeta, Z \rangle$, for all $Z \in \mathcal{Z}$. We obtain the second representation (4.6) of $\rho^{\alpha}(Z)$ by solving

(4.10)
$$\min_{X} \langle \mu, Z \rangle + \eta \langle \zeta, X \rangle$$

s.t. $X \succeq Z - \alpha$
 $X \succeq 0, X \in \mathcal{Z}.$

Suppose now that Ω is a finite probability space of n elements with vector of probabilities $P = (p_1, \ldots, p_n)$. Then we identify \mathcal{Z} and \mathcal{Z}^* with \mathbb{R}^n , thus obtaining that representation (4.10) is given by the following linear program:

(4.11)
$$\min_{X \in \mathbb{R}^n} \sum_{i=1}^n p_i \mu_i Z_i + \eta \sum_{i=1}^n p_i \zeta_i X_i$$

s.t. $X_i \ge Z_i - \alpha_i, \qquad 1 \le i \le n$
 $X_i \ge 0, \qquad 1 \le i \le n$

Example 4. Let $p \ge 1$, $\eta > 0$, $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, and $\alpha \in \mathcal{Z}$ be a threshold function. Consider the threshold exponential risk measure given by

(4.12)
$$\rho_e^{\alpha}(Z) = \mathbb{E}(Z) + \eta \inf_{\tau > 0} \tau \ln \mathbb{E}\left(e^{\tau^{-1}[Z-\alpha]_+}\right),$$

for every $Z \in \mathcal{Z}$. Define coherent risk measures $\rho(Z) = \mathbb{E}(Z)$ and $\vartheta(Z) = \inf_{\tau>0} \tau \ln \mathbb{E}\left(e^{\tau^{-1}Z}\right)$. As seen before, the set $\partial \rho(0)$ is a singleton consisting of the density μ with respect to the reference probability P. On the other hand, [32, ex. 6.17] shows that $\partial \vartheta(0) =$ $\{\zeta \in \mathcal{Z}^* \mid \mathbb{E}[\zeta W] \leq \ln \mathbb{E}[e^W], \forall W \in \mathcal{Z}\}$. Applying the representation theorem of coherent risk measures [32, thm. 6.4] to ϑ , we obtain the second representation (4.6) of $\rho_e^{\alpha}(Z)$ by solving

(4.13)

$$\begin{aligned}
\sup_{\zeta} \min_{X} \langle \mu, Z \rangle + \eta \langle \zeta, X \rangle \\
\text{s.t. } \mathbb{E}[\zeta W] \leq \ln \mathbb{E}[e^{W}], \quad \forall W \in \mathcal{Z} \\
X \succeq Z - \alpha \\
X \succeq 0, \ X \in \mathcal{Z}, \ \zeta \in \mathcal{Z}^{*}.
\end{aligned}$$

Suppose now that p = 1 and so $\mathcal{Z} = \mathcal{L}_1(\Omega, \mathcal{F}, P)$ and $\mathcal{Z}^* = \mathcal{L}_{\infty}(\Omega, \mathcal{F}, P)$. Let Ω be a finite probability space of *n* elements with vector of probabilities $P = (p_1, \ldots, p_n)$. Then we identify \mathcal{Z} and \mathcal{Z}^* with \mathbb{R}^n , thus obtaining that representation (4.13) is given by the following linear program:

(4.14)

$$\sup_{\zeta \in \mathbb{R}^{n}} \min_{X \in \mathbb{R}^{n}} \sum_{i=1}^{n} p_{i} \mu_{i} Z_{i} + \eta \sum_{i=1}^{n} p_{i} \zeta_{i} X_{i}$$

$$\text{s.t.} \sum_{i=1}^{n} p_{i} \zeta_{i} W_{i} \leq \ln \mathbb{E}[e^{W}], \quad \forall W \in \mathbb{R}^{n}$$

$$X_{i} \geq Z_{i} - \alpha_{i}, \qquad 1 \leq i \leq n$$

$$X_{i} \geq 0, \qquad 1 \leq i \leq n.$$

We can get around the infinitely many constraints in (4.14) by using a delayed constraint generation method. In this method at iteration k we have $\{W^1, \ldots, W^k\} \subset \mathbb{R}^n$ and solve the master problem

(4.15)

$$\sup_{\zeta \in \mathbb{R}^{n}} \min_{X \in \mathbb{R}^{n}} \sum_{i=1}^{n} p_{i} \mu_{i} Z_{i} + \eta \sum_{i=1}^{n} p_{i} \zeta_{i} X_{i}$$

$$\text{s.t.} \sum_{i=1}^{n} p_{i} \zeta_{i} W_{i}^{j} \leq \ln \mathbb{E}[e^{W^{j}}], \quad 1 \leq j \leq k$$

$$X_{i} \geq Z_{i} - \alpha_{i}, \qquad 1 \leq i \leq n$$

$$X_{i} \geq 0, \qquad 1 \leq i \leq n.$$

Let $\zeta^k \in \mathbb{R}^n$ be an optimal solution of (4.15) and solve the following problem:

(4.16)
$$\max_{W \in \mathbb{R}^n} \sum_{i=1}^n p_i \zeta_i^k W_i - \ln \mathbb{E}[e^W]$$

We can solve (4.16) by setting $f(W) = \sum_{i=1}^{n} p_i \zeta_i^k W_i - \ln \mathbb{E}[e^W]$ and looking at the points where $\partial f(W) / \partial W = 0$. If the solution to (4.16) is non positive then ζ^k is an optimal solution to (4.14) and we are done. On the other hand, if the solution to (4.16) is positive, then we pick a W^{k+1} such that $\sum_{i=1}^{n} p_i \zeta_i^k W_i^{k+1} - \ln \mathbb{E}[e^{W^{k+1}}] > 0$, form the extended set $\{W^1, \ldots, W^k, W^{k+1}\}$, and iterate by solving the master problem (4.15) on the extended set.

5. Dynamic Risk Measures, Time Consistency and Other Properties

In this section we lay out the foundation of our risk evaluation method based on TRM. The intention is to develop the risk-measures support for models suitable for risk-averse sequential decision making with TRM. For an in-depth view at these topics for coherent risk measures see [11,21,27,29,32].

In the remainder of the section we use the following definitions. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space with a filtration $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_T \subset \mathcal{F}$ and an adapted sequence of random variables $Z_t, t = 0, \ldots, T$. Define the spaces $\mathcal{Z}_t = \mathcal{L}_p(\Omega, \mathcal{F}_t, \mathcal{P}), p \in [1, +\infty), t =$ $0, \ldots, T$ and let $\mathcal{Z}_{t,T} = \mathcal{Z}_t \times \cdots \times \mathcal{Z}_T$. We assume that Z_0 is deterministic and so $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and the space \mathcal{Z}_0 can be associated to \mathbb{R} .

In this paper we consider random variables $Z \in \mathcal{Z}_t$ as stage-wise costs. Our goal is to define a way to evaluate the risk of the cost subsequence Z_t, \ldots, Z_T by using threshold risk measures. We take the first steps towards this by defining a threshold conditional risk measure based on the TRM and the coherent conditional risk measures developed by Ruszczyński and Shapiro in [29].

5.1. Conditional Risk Measures. In the classical setting of multistage stochastic optimization, the main tool used to formulate the corresponding dynamic programming equations is the concept of conditional expectation. Given two sigma algebras $\mathcal{F}_t \subset \mathcal{F}_{t+1}$, we let \mathcal{F}_t represent our knowledge when the expectation is evaluated at time t, and \mathcal{F}_{t+1} to represent all events under immediate future consideration. In this context, the conditional expectation can be viewed as a linear mapping from a space of \mathcal{F}_{t+1} -measurable functions into a space of \mathcal{F}_t -measurable functions. We use this mapping idea to extend the concept of conditional expectation to risk measures starting with coherent risk measures.

Definition 3. A one-step coherent conditional risk measure is a lower semicontinuous function $\rho_t : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ satisfying the following axioms:

- (A1') Convexity: $\rho_t (kZ + (1-k)Z') \leq k\rho_t(Z) + (1-k)\rho_t(Z')$, for all $Z, Z' \in \mathbb{Z}_{t+1}$ and all $k \in [0, 1]$;
- (A2') Monotonicity: If $Z, Z' \in \mathcal{Z}_{t+1}$ and $Z \preceq Z'$, then $\rho_t(Z) \preceq \rho_t(Z')$;
- (A3') Translation equivariance: If $W \in \mathcal{Z}_t$ and $Z \in \mathcal{Z}_{t+1}$, then $\rho_t(Z+W) = \rho_t(Z) + W$;
- (A4') Positive homogeneity: If $k \ge 0$ and $Z \in \mathcal{Z}_{t+1}$, then $\rho_t(kZ) = k\rho_t(Z)$.

We are ready now to extend the conditional expectation to threshold risk measures.

Definition 4. Let $\rho_t : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ and $\vartheta_t : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ be two one-step coherent conditional risk measures. If t = 0, then ρ_0 and ρ_0 are assumed to be real-valued. Let $\eta > 0$ and $\alpha_t \in \mathcal{Z}_{t+1}$ be a threshold function. We define a *one-step threshold conditional risk measure* as the function $\rho_t^{\alpha_t} : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ given by

(5.1)
$$\rho_t^{\alpha_t}(Z) = \varrho_t(Z) + \eta \vartheta_t([Z - \alpha]_+),$$

for every $Z \in \mathcal{Z}_{t+1}$.

Using conditional expectations we can obtain one-step conditional risk measures based on the examples given before.

Example 5. Given two sigma algebras $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ and corresponding spaces \mathcal{Z}_t and \mathcal{Z}_{t+1} , we define the one-step conditional mean upper semideviation from target $\alpha_t \in \mathcal{Z}_{t+1}$ by

(5.2)
$$\rho_t^{\alpha_t}(Z) = \mathbb{E}\left(Z \,|\, \mathcal{F}_t\right) + \eta \,\mathbb{E}\left(\left[Z - \alpha\right]_+ \,\big|\, \mathcal{F}_t\right),$$

for every $Z \in \mathcal{Z}_{t+1}$. It is easy to check that this risk measure satisfies all the axioms of the one-step threshold conditional risk measures.

Example 6. Given two sigma algebras $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ and corresponding spaces \mathcal{Z}_t and \mathcal{Z}_{t+1} , we define the *one-step threshold exponential risk measure* by

(5.3)
$$\rho_{e,t}^{\alpha_t}(Z) = \mathbb{E}\left(Z \mid \mathcal{F}_t\right) + \eta \inf_{\tau > 0} \tau \ln \mathbb{E}\left(e^{\tau^{-1}[Z-\alpha]_+} \mid \mathcal{F}_t\right),$$

for every $Z \in \mathcal{Z}_{t+1}$. It is easy to check that this risk measure satisfies all the axioms of the one-step threshold conditional risk measures.

The following theorems show some of the basic properties of the one-step threshold conditional risk measures. The proofs of these theorems are deferred to the appendix.

Theorem 8. Let $\rho_t^{\alpha_t} : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ be a one-step threshold conditional risk measure with threshold function $\alpha_t \in \mathcal{Z}_{t+1}$. Then for each $t = 0, \ldots, T-1$, the function ρ_t^{α} satisfies the following properties:

- $(T0') \rho_0^{\alpha_0}$ is a real-valued, lower semicontinuous and subdifferentiable risk measure;
- (T1') Convexity: $\rho_t^{\alpha_t}(kZ + (1-k)W) \leq k\rho_t^{\alpha_t}(Z) + (1-k)\rho_t^{\alpha_t}(W)$, for all $Z, W \in \mathbb{Z}_{t+1}$ and all $k \in [0, 1]$;
- (T2') Monotonicity: If $Z, W \in \mathbb{Z}_{t+1}$ and $Z \preceq W$, then $\rho_t^{\alpha_t}(Z) \preceq \rho_t^{\alpha_t}(W)$;
- (T3') Translation equivariance: If $W \in \mathbb{Z}_t$ and $Z \in \mathbb{Z}_{t+1}$, then $\rho_t^{\alpha}(Z + W) = \rho_t^{\alpha-W}(Z) + W;$
- (T4') Positive homogeneity: If k > 0 and $Z \in \mathbb{Z}_{t+1}$, then $\rho_t^{\alpha_t}(kZ) = k \rho_t^{\alpha_t/k}(Z)$.

Notice that in (T3') we are not requiring the random variable $\alpha - W$ to be nonnegative, therefore $\alpha - W$ is a threshold function only when $\alpha - W \succeq 0$.

The one-step threshold conditional risk measures also hold some extra properties regarding its threshold functions.

Theorem 9. Let $\alpha_t, \beta_t \in \mathbb{Z}_{t+1}$ be threshold functions. The one-step threshold conditional risk measure satisfies the following properties:

- (C1') Convexity of threshold functions: $\rho_t^{\lambda \alpha_t + (1-\lambda)\beta_t}(Z) \leq \lambda \rho_t^{\alpha_t}(Z) + (1-\lambda)\rho_t^{\beta_t}(Z),$ for all $Z \in \mathcal{Z}_{t+1}$ and all $\lambda \in [0,1]$;
- (C2') Monotonicity of threshold functions: if $\alpha_t \leq \beta_t$ and $Z \in \mathbb{Z}_{t+1}$, then $\rho_t^{\alpha_t}(Z) \succeq \rho_t^{\beta_t}(Z)$.

5.2. Dynamic Risk Measures. From this moment on we fix $\eta > 0$ and $T \in \mathbb{N}$. We consider sequences of threshold functions as elements of the set $\mathcal{Z}_{1,T}^+ = \mathcal{Z}_1^+ \times \cdots \times \mathcal{Z}_T^+$, where \mathcal{Z}_t^+ denotes the set of nonnegative random variables belonging to \mathcal{Z}_t , $t = 1, \ldots, T$. To ease our notation we use bold Greek letters to denote sequences of threshold functions in $\mathcal{Z}_{1,T}^+$. Notice that a *threshold sequence* $\boldsymbol{\alpha} = \{\alpha_t\}_{t=0}^{T-1} \in \mathcal{Z}_{1,T}^+$ is such that $\alpha_t \in \mathcal{Z}_{t+1}^+$, for every $t = 1, \ldots, T - 1$. Given two threshold sequences $\boldsymbol{\alpha} = \{\alpha_t\}_{t=0}^{T-1}$ and $\boldsymbol{\beta} = \{\beta_t\}_{t=0}^{T-1}$, we say that $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ if $\alpha_t \preceq \beta_t$, for every $t = 0, \ldots, T - 1$.

Definition 5. Let $\boldsymbol{\alpha} = \{\alpha_t\}_{t=0}^{T-1} \in \mathcal{Z}_{1,T}^+$ and let $\{\rho_t^{\alpha_t}\}_{t=0}^{T-1}$ be an adapted sequence of one-step threshold conditional risk measures. A threshold dynamic risk measure (or a dynamic risk measure for short) is a sequence of mappings $\{\rho_{t,T}^{\boldsymbol{\alpha}}\}_{t=0}^{T}$, where

$$\rho_{t,T}^{\boldsymbol{\alpha}} : \mathcal{Z}_{t,T} \to \mathcal{Z}_t \text{ is given by}
(5.4)
\rho_{t,T}^{\boldsymbol{\alpha}}(Z_t, \dots, Z_T) = Z_t + \rho_t^{\alpha_t} \left(Z_{t+1} + \rho_{t+1}^{\alpha_{t+1}} \left(Z_{t+2} + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(Z_{T-1} + \rho_{T-1}^{\alpha_{T-1}}(Z_T) \right) \cdots \right) \right),
t = 0, \dots, T.$$

The following theorem states some of the most relevant properties of the mappings defined in (5.4).

Theorem 10. Let $\left\{\rho_{t,T}^{\boldsymbol{\alpha}}\right\}_{t=0}^{T}$ be a threshold dynamic risk measure. Then $\rho_{0,T}^{\boldsymbol{\alpha}}$ is realvalued and lower semicontinuous. In addition, the sequence of mappings $\left\{\rho_{t,T}^{\boldsymbol{\alpha}}\right\}_{t=0}^{T}$ satisfy the following properties:

- (DR1) Convexity in $\mathcal{Z}_{t,T}$: $\rho_{t,T}^{\boldsymbol{\alpha}}(kZ + (1-k)W) \leq k\rho_{t,T}^{\boldsymbol{\alpha}}(Z) + (1-k)\rho_{t,T}^{\boldsymbol{\alpha}}(W)$, for all $Z, W \in \mathcal{Z}_{t,T}$ and all $k \in [0,1]$;
- (DR2) Monotonicity in $\mathcal{Z}_{t,T}$: If $Z, W \in \mathcal{Z}_{t,T}$ and $Z \preceq W$, then $\rho_{t,T}^{\boldsymbol{\alpha}}(Z) \preceq \rho_{t,T}^{\boldsymbol{\alpha}}(W)$; (DR3) Convexity in threshold sequences: $\rho_{t,T}^{\lambda \boldsymbol{\alpha} + (1-\lambda)\boldsymbol{\beta}}(Z) \preceq \lambda \rho_{t,T}^{\boldsymbol{\alpha}}(Z) + (1-\lambda)\rho_{t,T}^{\boldsymbol{\beta}}(Z)$, for all $Z \in \mathcal{Z}_{t,T}$, all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{Z}_{1,T}^+$, and all $\lambda \in [0, 1]$;
- (DR4) Monotonicity in threshold sequences: If $Z \in \mathcal{Z}_{t,T}$, $\alpha, \beta \in \mathcal{Z}_{1,T}^+$, and $\alpha \preceq \beta$ then $\rho_{t,T}^{\boldsymbol{\alpha}}(Z) \succeq \rho_{t,T}^{\boldsymbol{\beta}}(Z).$

Proof. We start by showing that $\rho_{0,T}^{\alpha}$ is real-valued.

$$\rho_{0,T}^{\boldsymbol{\alpha}}(Z_0,\ldots,Z_T) = Z_0 + \rho_0^{\alpha_0} \left(Z_1 + \rho_1^{\alpha_1} \left(Z_2 + \cdots + \rho_{T-2}^{\alpha_{T-2}} \left(Z_{T-1} + \rho_{T-1}^{\alpha_{T-1}} (Z_T) \right) \cdots \right) \right)$$

= $Z_0 + \rho_0^{\alpha_0} \left(\rho_{1,T}^{\boldsymbol{\alpha}}(Z_1,\ldots,Z_T) \right)$
< ∞ ,

where the last inequality holds because Z_0 is deterministic (since $\mathcal{F}_0 = \{\Omega, \emptyset\}$) and, by definition, $\rho_0^{\alpha_0}$ is a threshold risk measure which is real-valued by Theorem 4. The function $\rho_{0,T}^{\alpha}$ is by definition a composition of lower semicontinuous functions and as such it is a lower semicontinuous function itself.

By applying recursively the convexity and monotonicity properties (T1')-(T2') of the one-step threshold conditional risk measures (Theorem 8) we can see that properties (DR1) and (DR2) hold. Similarly, the recursive application of properties (C1')-(C2')from Theorem 9, show that properties (DR3) and (DR4) hold. \square

Our use of the term dynamic risk measure is based on the work presented in [7. 12, 13, 15, 25, 27]. In light of Theorem (10) we say that the threshold dynamic risk measure $\left\{\rho_{t,T}^{\alpha}\right\}_{t=0}^{T}$ is proper, lower semicontinuous, convex, and monotone. Now we can leverage the theory and concepts already defined for dynamic risk measures to give a solid foundation to our risk evaluation method. In the rest of this section we discuss some important properties of threshold dynamic risk measures.

Theorem 11. Let $0 \leq t < r \leq T$ and $Z, \tilde{Z} \in \mathcal{Z}_{t,T}$ be such that $Z_i \preceq \tilde{Z}_i, i = t, \ldots, r-1$. Then $\rho_{r,T}^{\alpha}(Z_r, \ldots, Z_T) \preceq \rho_{r,T}^{\alpha}(\tilde{Z}_r, \ldots, \tilde{Z}_T)$ implies that $\rho_{t,T}^{\alpha}(Z_t, \ldots, Z_T) \preceq \rho_{t,T}^{\alpha}(\tilde{Z}_t, \ldots, \tilde{Z}_T)$.

Proof. By the monotonicity property of $\rho_{t,T}^{\alpha}$ we get that for any r that satisfies $0 \le t < r \le T$:

$$\rho_{t,T}^{\boldsymbol{\alpha}}(Z_{t},\ldots,Z_{T}) = Z_{t} + \rho_{t}^{a_{t}} \left(Z_{t+1} + \rho_{t+1}^{a_{t+1}} \left(Z_{t+2} + \cdots + \rho_{r-2}^{a_{r-2}} \left(Z_{r-1} + \rho_{r-1}^{a_{r-1}} \left(\rho_{r,T}^{\boldsymbol{\alpha}}(Z_{r},\ldots,Z_{T}) \right) \right) \cdots \right) \right) \right) \\
\leq Z_{t} + \rho_{t}^{a_{t}} \left(Z_{t+1} + \rho_{t+1}^{a_{t+1}} \left(Z_{t+2} + \cdots + \rho_{r-2}^{a_{r-2}} \left(Z_{r-1} + \rho_{r-1}^{a_{r-1}} \left(\rho_{r,T}^{\boldsymbol{\alpha}}(\tilde{Z}_{r},\ldots,\tilde{Z}_{T}) \right) \right) \cdots \right) \right) \\
\leq \tilde{Z}_{t} + \rho_{t}^{a_{t}} \left(\tilde{Z}_{t+1} + \rho_{t+1}^{a_{t+1}} \left(\tilde{Z}_{t+2} + \cdots + \rho_{r-2}^{a_{r-2}} \left(\tilde{Z}_{r-1} + \rho_{r-1}^{a_{r-1}} \left(\rho_{r,T}^{\boldsymbol{\alpha}}(\tilde{Z}_{r},\ldots,\tilde{Z}_{T}) \right) \right) \cdots \right) \right) \\
= \rho_{t,T}^{\boldsymbol{\alpha}} (\tilde{Z}_{t},\ldots,\tilde{Z}_{T}).$$

We call the property described in Theorem 11 the *time consistency* of the dynamic risk measure $\left\{\rho_{t,T}^{\alpha}\right\}_{t=0}^{T}$ and it is adapted from similar concepts introduced in [25, 27]. Time consistency means that if the sequence Z is not more than \tilde{Z} from time t to r-1 and Z is perceived as less risky than \tilde{Z} from the point of view of some future time r, then Z is less risky than \tilde{Z} . Thanks to time consistency our dynamic risk measure does not need to backtrack its decisions when it is computed in a recursive manner from tail to head. This is a key property that allows the development of Bellman-type equations for the evaluation of our threshold dynamic risk measure.

For a threshold dynamic risk measure $\left\{\rho_{t,T}^{\alpha}\right\}_{t=0}^{T}$ we define a broader family of *threshold conditional risk measures*, by setting

(5.5)
$$\rho_{t,r}^{\alpha}(Z_t, \dots, Z_r) = \rho_{t,T}^{\alpha}(Z_t, \dots, Z_r, 0, \dots, 0), \quad 0 \le t \le r \le T.$$

This extension is based on similar concepts discussed in [27] where a similar extension is performed for coherent conditional risk measures.

Another important property of dynamic risk measures is the local property, discussed in detail in [7, 18, 19, 27].

Definition 6. A threshold (or coherent) conditional risk measure $\rho_{t,r}$, $0 \le t \le r \le T$, has the local property if for all sequences $Z \in \mathcal{Z}_{t,r}$ and all events $\mathfrak{B} \in \mathcal{F}_t$ we have

(5.6)
$$\rho_{t,r}(\mathbb{1}_{\mathfrak{B}}Z) = \mathbb{1}_{\mathfrak{B}}\rho_{t,r}(Z).$$

We say that a threshold (or coherent) dynamic risk measure $\{\rho_t\}_{t=0}^T$ has the *local property* if for every $0 \le t < r \le T$, the conditional risk measure $\rho_{t,r}$ has the local property.

The local property is desired because it avoids the counterintuitive situation where an event that does not happen at time t influences the measure of risk at a later time. In other words, if $\mathfrak{B} \in \mathcal{F}_t$ has measure zero (i.e \mathfrak{B} does not happen at time t) then

$$\rho_{t,r}^{\boldsymbol{\alpha}}(\mathbb{1}_{\mathfrak{B}}Z) = \mathbb{1}_{\mathfrak{B}}\rho_{t,r}^{\boldsymbol{\alpha}}(Z) = 0.$$

It is known that coherent dynamic risk measures have the local property, see [19,27,29]. Using this result we can show that threshold dynamic risk measures also have property (5.6).

Theorem 12. Let $\eta > 0$, $\{\varrho_t\}_{t=0}^T$, $\{\vartheta_t\}_{t=0}^T$ be sequences of one-step coherent conditional risk measures such that ϱ_0 and ϑ_0 are real-valued, and let $\boldsymbol{\alpha} = \{\alpha_t\}_{t=0}^{T-1} \in \mathcal{Z}_{1,T}^+$.

Let $\{\rho_t^{\alpha}\}_{t=0}^{T-1}$ be the threshold dynamic risk measure obtained from the one-step threshold conditional risk measures defined by $\rho_t^{\alpha_t}(Z) = \varrho_t(Z) + \eta \vartheta_t[Z - \alpha_t]_+, \ \forall Z \in \mathcal{Z}_{t+1}.$

Then for all $0 \leq k \leq r \leq T$, all events $\mathfrak{B} \in \mathcal{F}_k$, and all $Z \in \mathcal{Z}_{k,r}$:

$$\rho_{k,r}^{\boldsymbol{\alpha}}(\mathbb{1}_{\mathfrak{B}}Z) = \mathbb{1}_{\mathfrak{B}}\rho_{k,r}^{\boldsymbol{\alpha}}(Z).$$

In other words, the threshold dynamic risk measure $\{\rho_t^{\alpha}\}_{t=0}^{T-1}$ has the local property.

Proof. For $\alpha \in \mathbb{Z}_k^+$, $\mathfrak{B} \in \mathbb{F}_k$, and $Z \in \mathbb{Z}_{k+1}$, the threshold conditional risk measure ρ_k^{α} satisfies:

(5.7)

$$\rho_{k}^{\alpha}(\mathbb{1}_{\mathfrak{B}}Z) = \varrho_{k}(\mathbb{1}_{\mathfrak{B}}Z) + \eta \vartheta_{k}([\mathbb{1}_{\mathfrak{B}}Z - \alpha]_{+}) \\
= \varrho_{k}(\mathbb{1}_{\mathfrak{B}}Z) + \eta \vartheta_{k}(\mathbb{1}_{\mathfrak{B}}[Z - \alpha]_{+}) \\
= \mathbb{1}_{\mathfrak{B}}\left(\varrho_{k}(Z) + \eta \vartheta_{k}\left([Z - \alpha]_{+}\right)\right) \\
= \mathbb{1}_{\mathfrak{B}}\rho_{k}^{\alpha}(Z),$$

where in the second equality we used Theorem 2. Applying (5.7) recursively we obtain that

$$\begin{split} \rho_{t,r}^{\alpha}(\mathbb{1}_{\mathfrak{B}}Z) &= \mathbb{1}_{\mathfrak{B}}Z_{t} + \rho_{t}^{\alpha_{t}} \left(\mathbb{1}_{\mathfrak{B}}Z_{t+1} + \rho_{t+1}^{\alpha_{t+1}} \left(\mathbb{1}_{\mathfrak{B}}Z_{t+2} + \dots + \rho_{r-2}^{\alpha_{r-2}} \left(\mathbb{1}_{\mathfrak{B}}Z_{r-1} + \rho_{r-1}^{\alpha_{r-1}} (\mathbb{1}_{\mathfrak{B}}Z_{r}) \right) \cdots \right) \right) \\ &= \mathbb{1}_{\mathfrak{B}}Z_{t} + \rho_{t}^{\alpha_{t}} \left(\mathbb{1}_{\mathfrak{B}}Z_{t+1} + \rho_{t+1}^{\alpha_{t+1}} \left(\mathbb{1}_{\mathfrak{B}}Z_{t+2} + \dots + \rho_{r-2}^{\alpha_{r-2}} \left(\mathbb{1}_{\mathfrak{B}} \left[Z_{r-1} + \rho_{r-1}^{\alpha_{r-1}} (Z_{r}) \right] \right) \cdots \right) \right) \right) \\ &\vdots \\ &= \mathbb{1}_{\mathfrak{B}} \left[Z_{t} + \rho_{t}^{\alpha_{t}} \left(Z_{t+1} + \rho_{t+1}^{\alpha_{t+1}} \left(Z_{t+2} + \dots + \rho_{r-2}^{\alpha_{r-2}} \left(Z_{r-1} + \rho_{r-1}^{\alpha_{r-1}} (Z_{r}) \right) \cdots \right) \right) \right) \right] \\ &= \mathbb{1}_{\mathfrak{B}} \rho_{t,r}^{\alpha}(Z). \end{split}$$

6. FINITE HORIZON RISK-AVERSE MARKOV MODEL AND BELLMAN'S EQUATION

In this section we define our finite horizon risk-averse Markov model and describe a Bellman-type equation for its recursive solution. The Bellman-type equation plays a fundamental role in the development of efficient methods for the numerical solution of our risk-averse Markov models. We start first by building our Markov model.

6.1. A Risk-Averse Markov Model. Fix $p \in [1, +\infty)$ be fixed. Consider the Markov decision process introduced in section 2. Each policy $\pi = \{\pi_0, \ldots, \pi_T\}$ results in a cost sequence $Z_0^{\pi}, \ldots, Z_T^{\pi}$, where for each $t = 0, \ldots, T - 1$, $Z_t^{\pi} = c_t(s_t, \pi_t(s_t))$, with $Z_T^{\pi} = c_T(s_T)$. Given a policy π , we can interpret each Z_t^{π} as a random variable $Z_t^{\pi} : S \to \mathbb{R}$.

Fix the first state $s_0 \in \mathcal{S}$ and let $\Omega = \mathcal{S}^{T+1}$. For each $t = 0, \ldots, T$, let \mathcal{F}_t be the σ -algebra on Ω generated by policies π starting at s_0 and having the same random Z^{π} outcomes up to time t. We assume that the state space \mathcal{S} is connected in the sense that $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We call $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_T$ the *canonical information filtration* on Ω . Let $\mathcal{F} = \mathcal{F}_T$ and P be the probability measure on (Ω, \mathcal{F}) obtained from the underlying Markov process as described in section 2. We restrict ourselves to random variables $Z_t^{\pi} \in \mathcal{L}_p(\Omega, \mathcal{F}_t, P)$.

To simplify our exposition from now on we drop the π from the notation of the random variable Z_t^{π} . We do this in such a way that at any moment it is clear from context to which policy π the random variable Z_t corresponds.

Let $\mathcal{Z}_t = \mathcal{L}_p(\Omega, \mathcal{F}_t, P), t = 0, \dots, T$, and consider a threshold sequence $\boldsymbol{\alpha} = \{\alpha_t\}_{t=0}^{T-1} \in \mathcal{Z}_{1,T}^+$. We evaluate the risk of a policy π by applying the threshold dynamic risk measure $\left\{\rho_{t,T}^{\boldsymbol{\alpha}}\right\}_{t=0}^{T-1}$ to its random cost sequence Z_0, \dots, Z_T . By definition this is:

$$\begin{aligned} &(6.1) \\ &R(s_0,\pi) = \rho_{0,T}^{\boldsymbol{\alpha}}(Z_0,\dots,Z_T) \\ &= Z_0 + \rho_0^{\alpha_0} \left(Z_1 + \rho_1^{\alpha_1} \left(Z_2 + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(Z_{T-1} + \rho_{T-1}^{\alpha_{T-1}} (Z_T) \right) \cdots \right) \right) \right) \\ &= c_0(s_0,\pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1,\pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \right) \right) \right) \\ &= c_0(s_0,\pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1,\pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \right) \right) \right) \\ &= c_0(s_0,\pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1,\pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \right) \right) \right) \\ &= c_0(s_0,\pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1,\pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \right) \right) \\ &= c_0(s_0,\pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1,\pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \right) \right) \\ &= c_0(s_0,\pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1,\pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \right) \right) \\ &= c_0(s_0,\pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1,\pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1}) \right) \right) \right) \\ &= c_0(s_0,\pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1,\pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1}) \right) \right) \right) \\ &= c_0(s_0,\pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1,\pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1}) \right) \right)$$

where $\rho_t^{\alpha_t} : \mathcal{Z}_{t+1} \to \mathcal{Z}_t, t = 0, \dots, T-1$, is a one-step threshold conditional risk measure. This is what brings us to the core of the difficulty in evaluating (6.1): at time t the function $\rho_t^{\alpha_t}$ really depends on the entire history of the process up to time t, but to have any hopes of obtaining a recursive method of solving the risk-averse dynamic problem we should have a way to represent $\rho_t^{\alpha_t}$ as functions of \mathcal{S} . To overcome this we follow the decomposition method presented in [27] and construct a new conditional risk measure that takes as argument measurable functions on the state space \mathcal{S} instead of the probability space Ω .

Let P_0 be a fixed probability measure on $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$, where $\mathcal{B}_{\mathcal{S}}$ denotes the usual Borel σ -algebra on \mathcal{S} . Let $\mathcal{V} = \mathcal{L}_p(\mathcal{S}, \mathcal{B}, P_0), \mathcal{V}^* = \mathcal{L}_q(\mathcal{S}, \mathcal{B}, P_0)$ with $p, q \in [1, +\infty]$ such that 1/p + 1/q = 1. Let \mathcal{V}_+ be the set of nonnegative random variables belonging to \mathcal{V} . We saw in section 3 that the spaces $\mathcal{V}, \mathcal{V}^*$ are conjugate dual paired with the scalar product

$$\langle y, v \rangle = \int_{\mathcal{S}} y(s)v(s)dP_0(s),$$

for all $v \in \mathcal{V}, y \in \mathcal{V}^*$. Let $\mathcal{M} \subseteq \mathcal{V}^*$ be given by

(6.2)
$$\mathcal{M} = \left\{ m \in \mathcal{V}^* \, \middle| \, \int_{\mathcal{S}} m(s) dP_0(s) = 1 \text{ and } m(s') \ge 0, \, \forall s' \in \mathcal{S} \right\}.$$

By definition each element $m \in \mathcal{M}$ gives rise to a probability measure $m dP_0$ on $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$ given in reference to the fixed probability P_0 and having m as its density.

In our application to dynamic programming the set \mathcal{V} plays the role of the set of value functions while \mathcal{M} could be seen as the set of densities over the states arising from the state-action pairs. The main purpose of the next few paragraphs and definitions is to formalize this in a way conducive to a Bellman-type equation. For this reason at this moment we add one key assumption:

From now on we assume that the controlled kernels Q_t take values in the set \mathcal{M} . Thus, the set \mathcal{M} contains all the densities imposed by the state-action pairs on the state space $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$.

We follow [27] and give two important definitions. For every $s \in S$, let A(s) denote the action set of s, and let \mathcal{A} denote the action space of the controlled Markov process $\{s_t\}_{t=0}^T$.

Definition 7. A measurable functional $\sigma : \mathcal{V} \times \mathcal{V}_+ \times \mathcal{S} \times \mathcal{M} \to \mathbb{R}$ is a threshold risk transition mapping associated with the controlled kernel $Q : \operatorname{graph}(A) \to \mathcal{M}$ if

- (i) For every $\alpha \in \mathcal{V}_+$, $s \in \mathcal{S}$, and $m \in \mathcal{M}$, the functional $v \mapsto \sigma(v, \alpha, s, m)$ is a threshold risk measure on \mathcal{V} with threshold function α ;
- (ii) For every $v \in \mathcal{V}$, every $\alpha \in \mathcal{V}_+$, and every measurable selection $a(\cdot)$ of $A(\cdot)$ the function $s \mapsto \sigma(v, \alpha, s, Q(s, a))$ is an element of \mathcal{V} .

Definition 8. A one-step threshold conditional risk measure $\rho_t^{\alpha_t} : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ is a threshold Markov risk measure with respect to the controlled Markov process $\{s_t\}_{t=0}^T$, if there exist $\overline{\alpha}_t \in \mathcal{V}_+$ and a threshold risk transition mapping $\sigma_t : \mathcal{V} \times \mathcal{V}_+ \times \mathcal{S} \times \mathcal{M} \to \mathbb{R}$ such that for all $v \in \mathcal{V}$ and all measurable $a_t \in A(s_t)$ we have

(6.3)
$$\rho_t^{\alpha_t}(v(s_{t+1})) = \sigma_t(v, \overline{\alpha}_t, s_t, Q_t(s_t, a_t)).$$

Example 7. Now we use the mean upper semideviation from target to construct a threshold Markov risk measure. Define $\sigma_t : \mathcal{V} \times \mathcal{V}_+ \times \mathcal{S} \times \mathcal{M} \to \mathbb{R}$ by

(6.4)
$$\sigma_t(v,\alpha,s,m) = \langle v,m \rangle + \eta \langle (v-\alpha)_+,m \rangle.$$

It is not difficult to see that σ_t is a threshold risk transition mapping. Let ρ_t^{α} be the one-step conditional mean upper semideviation from target defined in example 5. The following proposition shows that ρ_t^{α} is a threshold Markov risk measure associated to σ_t .

Proposition 13. Let $\alpha_t \in \mathcal{V}_+$, $s_t \in \mathcal{S}$, and $a_t \in A_t(s)$ be a measurable selection, $t = 0, \ldots, T-1$. Let σ_t be as in (6.4) and let ρ_t^{α} be the one-step conditional mean upper semideviation from target defined in example 5. Then for all $v \in \mathcal{V}$ we have:

$$\rho_t^{\alpha_t}(v(s_{t+1})) = \sigma_t(v, \alpha_t, s_t, Q_t(s_t, a_t)).$$

Proof. Let $\mathbb{E}_{Q_t(s_t,a_t)}$ denote the expectation with respect to the probability measure $Q_t(s_t, a_t)dP_0$ defined by the density $Q_t(s_t, a_t) \in \mathcal{M}$. Notice that by definition $\mathbb{E}_{Q_t(s_t, a_t)}(v) = \mathbb{E}(v(s_{t+1}) | \mathcal{F}_t)$, for any $v \in \mathcal{V}$. Then

$$\sigma_t(v, \alpha_t, s_t, Q_t(s_t, a_t)) = \langle v, Q_t(s_t, a_t) \rangle + \eta \langle [v - \alpha_t]_+, Q_t(s_t, a_t) \rangle$$

= $\mathbb{E}_{Q_t(s_t, a_t)}(v) + \eta \mathbb{E}_{Q_t(s_t, a_t)}([v - \alpha_t]_+)$
= $\mathbb{E} (v(s_{t+1}) | \mathcal{F}_t) + \eta \mathbb{E} ([v(s_{t+1}) - \alpha_t]_+ | \mathcal{F}_t)$
= $\rho_t^{\alpha_t}(v(s_{t+1})).$

The following theorem uses Theorem 7, i.e. the second representation theorem of threshold risk measures, to give an easy to calculate representation for a threshold Markov risk measure. This is key to the development of Bellman-type equations to solve our main risk-averse problem with threshold risk measures.

Theorem 14. Consider a controlled Markov process with state space S, action space A, action sets A_t , controlled kernels Q_t : graph $(A) \to M$, and cost functions c_t , for $t = 0, \ldots, T - 1$. Let $\sigma_t : \mathcal{V} \times \mathcal{V}_+ \times S \times \mathcal{M} \to \mathbb{R}$, $t = 0, \ldots, T - 1$, be threshold risk transition mappings associated to the controlled Markov process. Moreover, suppose that for every $\alpha \in \mathcal{V}_+$, every $s \in S$, and every $m \in \mathcal{M}$ the functional $v \mapsto \sigma_t(v, \alpha, s, m)$, $t = 0, \ldots, T - 1$, is lower semicontinuous.

Let $\overline{\alpha}_t \in \mathcal{V}_+$ and $\rho_t^{\alpha_t} : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$, $t = 0, \ldots, T-1$, be threshold Markov risk measures with respect to the controlled Markov process such that

(6.5)
$$\rho_t^{\alpha_t}(v(s_{t+1})) = \sigma_t(v, \overline{\alpha}_t, s_t, Q_t(s_t, a_t)),$$

for all measurable selections $a_t \in A(s_t)$.

Then there exist closed convex-valued multifunctions $\mathcal{G}_t : \mathcal{S} \times \mathcal{M} \rightrightarrows \mathcal{M} \times \mathcal{M}, t = 0, \ldots, T-1$, such that $\rho_t^{\alpha_t}(v(s_{t+1}))$ is obtained by solving the following optimization problem on random variables

(6.6)
$$\sup_{\mu,\zeta} \min_{x} \langle \mu, v \rangle + \eta \langle \zeta, x \rangle$$
$$s.t. \ x \succeq v - \overline{\alpha}_{t}$$
$$x \succeq 0, \ x \in \mathcal{V}$$
$$(\mu,\zeta) \in \mathcal{G}_{t} \left(s_{t}, Q_{t}(s_{t}, a_{t}) \right).$$

If in addition, the functional $\sigma_t(\cdot, \alpha, s, m)$ is continuous, then the multifunction \mathcal{G}_t is bounded, $t = 0, \ldots, T - 1$. Moreover, if $p \in [1, +\infty)$, then $\sigma_t(\cdot, \alpha, s, m)$ is continuous and \mathcal{G}_t is weakly*-compact, $t = 0, \ldots, T - 1$. In this case we can replace the "sup" in (6.6) by the "max" operation.

Proof. Let $s \in \mathcal{S}$ and $m \in \mathcal{M}$ be a measurable selection. By assumption, the functional $v \mapsto \sigma_t(v, \alpha, s, m)$ is a lower semicontinuous threshold risk measure. Then by the second representation theorem of TRM (i.e. Theorem 7) there are closed convex sets $G_t^1(s, m), G_t^2(s, m) \subset \mathcal{M}$ such that $\sigma_t(v, \alpha, s, m)$ is given by

(6.7)
$$\sup_{\mu,\zeta} \min_{x} \langle \mu, v \rangle + \eta \langle \zeta, x \rangle$$
$$\operatorname{s.t.} x \succeq v - \alpha$$
$$x \succeq 0, x \in \mathcal{V}$$
$$\mu \in G_t^1(s, m), \ \zeta \in G_t^2(s, m).$$

If in addition the functional $\sigma_t(\cdot, \alpha, s, m)$ is continuous, then we can conclude that the sets $G_t^i(s, m)$, i = 1, 2 are bounded. If $p \in [1, +\infty)$, then the convexity and monotonicity properties of threshold risk measures guarantee that $\sigma_t(\cdot, \alpha, s, m)$ is continuous. Therefore $G_t^i(s, m)$, i = 1, 2 are weakly*-compact.

Define the functions $\mathcal{G}_t : \mathcal{S} \times \mathcal{M} \rightrightarrows \mathcal{M} \times \mathcal{M}, t = 0, \dots, T-1$ by $\mathcal{G}_t(s,m) = G_t^1(s,m) \times G_t^2(s,m)$. By the definition of threshold Markov risk measure, for any $s_t \in \mathcal{S}$ and any measurable selection $a_t \in A(s_t)$ we have that $\rho_t^{\alpha_t}(v(s_{t+1})) = \sigma_t(v, \overline{\alpha}_t, s_t, Q(s_t, a_t))$, thus completing the proof by applying (6.7) and the definition of \mathcal{G}_t .

Definition 9. The multifunction $\mathcal{J}_t : \mathcal{S} \times \mathcal{A} \rightrightarrows \mathcal{M} \times \mathcal{M}$ given by the composition

(6.8)
$$\mathcal{J}_t(s_t, a_t) = \mathcal{G}_t(s_t, Q_t(s_t, a_t))$$

is called a *threshold controlled multikernel* associated with the controlled Markov process $\{s_t\}$ and the conditional threshold risk mapping $\rho_t^{\alpha_t}$, $t = 0, \ldots, T - 1$.

Fix $\overline{\alpha}_t \in \mathcal{V}_+$, $t = 0, \ldots, T - 1$. Let $\Psi_t : \mathcal{V} \times \operatorname{graph}(A_t) \to \mathbb{R}$, $t = 0, \ldots, T - 1$, be defined by

(6.9)
$$\Psi_t(v, s_t, a_t) = \sigma_t(v, \overline{\alpha}_t, s_t, Q(s_t, a_t))$$

The continuity property of Ψ_t discussed below plays a key role in the main proof for the Bellman-type recursion of threshold risk measures.

Theorem 15. If $Q_t(s_t, \cdot)$ is continuous and $\mathcal{G}_t(s_t, \cdot)$ is lower semicontinuous, then the function $\Psi_t(v, s_t, \cdot)$ is lower semicontinuous.

Proof. Since $Q_t(s_t, \cdot)$ is a continuous function, the composition $\mathcal{J}_t(s_t, \cdot) = \mathcal{G}_t(s_t, Q_t(s_t, \cdot))$ inherits the lower semicontinuity property of $\mathcal{G}_t(s_t, Q_t(s_t, \cdot))$. For every $v \in \mathcal{V}$, define the function $\psi_v : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ given by

$$\psi_v(\mu,\zeta) = \min_x \langle \mu, v \rangle + \eta \langle \zeta, x \rangle$$

s.t. $x \succeq v - \overline{\alpha}_t$
 $x \succ 0, x \in \mathcal{V}.$

Clearly ψ_v is continuous on $\mathcal{M} \times \mathcal{M}$. Then by [6, thm. 1.4.16], the function

$$\Psi_t(v, s_t, \cdot) = \sup_{(\mu, \zeta) \in \mathcal{G}_t(s_t, Q_t(s_t, \cdot))} \psi_v(\mu, \zeta)$$

is lower semicontinuous.

6.2. Bellman-Type Equation for Finite Horizon Threshold Risk Measures. With these definitions and results at hand we can state our risk-averse optimization problem and a solution with Bellman-type equation. Fix $T \ge 1$, an initial state $s_0 \in S$, and a threshold sequence α . Our main goal is to find the policy π that minimizes the risk evaluation $R(s_0, \pi)$ given in (6.1). That is, we want to solve

(6.10)
$$\min_{\pi \in \Pi} R(s_0, \pi),$$

where Π denotes the set of admissible policies and

$$R(s_0,\pi) = c_0(s_0,\pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1,\pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-1}} \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_{T-1}) + \rho_{T-1}^{\alpha_{T-1}} \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_{T-1}) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_{T-1}) + \rho_{T-1}^{\alpha_{T-1}} \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_{T-1}) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_{T-1}) + \rho_{T-1}^{\alpha_{T-1}} \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_{T-1}) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_{T-1}) + \rho_{T-1}^{\alpha_{T-1}} \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_{T-1}) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_{T-1}) + \rho_{T-1}^{\alpha_{T-1}} \right) \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_{T-1}) + \rho_{T-1}^{\alpha_{T-1}} \right) + \rho_{T-1}^{\alpha_{T-1}} \left(c_2(s_2,\pi_{T-1}) + \rho_{T-1}^{\alpha_{T-1}} \right) \right)$$

We call (6.10) the *finite horizon threshold risk-averse problem*. The following theorem gives a Bellman-type equation that solves problem (6.10) for the case where our decomposition method applies.

Theorem 16. Consider the risk-averse problem (6.10) together with all mathematical constructions and definitions given in this section. Assume that the following conditions are satisfied:

(i) For every $s \in S$ the transition kernels $Q_t(s, \cdot), t = 0, \ldots, T-1$, are continuous;

- (ii) The threshold conditional risk measures $\rho_t^{\alpha_t}$, $t = 0, \ldots, T-1$ are threshold Markov with $\overline{\alpha}_t \in \mathcal{V}_+$ and such that for every $s \in \mathcal{S}$ the multifunctions $\mathcal{G}_t(s, \cdot)$ are lower semicontinuous;
- (iii) For all measurable selections $a_t(\cdot) \in A_t(\cdot)$, the functions $s \mapsto c_t(s, a_t(s))$, $t = 0, \ldots, T-1$, and $c_T(\cdot)$ are elements of \mathcal{V} ;
- (iv) For every $s \in S$ the functions $c_t(s, \cdot)$, $t = 0, \ldots, T$, are lower semicontinuous;
- (v) For every $s \in S$ the sets $A_t(s), t = 0, \ldots, T-1$, are compact

Then problem (6.10) has an optimal solution and its optimal value $v_0(s_0)$ is the solution of the following Bellman-type equations:

(6.11)
$$v_T(s) = c_T(s), \quad s \in \mathcal{S},$$

(6.12)
$$v_t(s) = \min_{a \in A_t(s)} \left\{ c_t(s, a) + \sigma_t \left(v_{t+1}, \overline{\alpha}_t, s, Q_t \left(s, a \right) \right) \right\}, \ s \in \mathcal{S}, \ t = T - 1, \dots, 0,$$

where for $t = 0, \ldots, T - 1$ we have

(6.13)

$$\sigma_t(v, \overline{\alpha}_t, s, Q_t(s, a)) = \sup_{\mu, \zeta} \min_x \langle \mu, v \rangle + \eta \langle \zeta, x \rangle$$

$$s.t. \ x \succeq v - \overline{\alpha}_t$$

$$x \succeq 0, x \in \mathcal{V}$$

$$(\mu, \zeta) \in \mathcal{G}_t(s, Q_t(s, a)).$$

Moreover, an optimal Markov policy $\hat{\pi} = \{\hat{\pi}_0, \dots, \hat{\pi}_{T-1}\}$ exists and satisfies the equations:

(6.14)
$$\hat{\pi}_t(s) \in \underset{a \in A_t(s)}{\arg\min} \{ c_t(s, a) + \sigma_t(v_{t+1}, \overline{\alpha}_t, s, Q_t(s, a)) \}, s \in \mathcal{S}, t = T - 1, \dots, 0.$$

Conversely, any measurable solution of equations (6.11)–(6.14) is an optimal Markov policy $\hat{\pi}$.

Proof. Applying the monotonicity condition (T2') to ρ^{α_t} , $t = 0, \ldots, T-1$, problem (6.10) can be written as:

$$\min_{\pi_{0},\dots,\pi_{T-1}} \left\{ c_{0}(s_{0},\pi_{0}(s_{0})) + \rho_{0}^{\alpha_{0}} \left(c_{1}(s_{1},\pi_{1}(s_{1})) + \rho_{1}^{\alpha_{1}} \left(c_{2}(s_{2},\pi_{2}(s_{2})) + \cdots \right) \right) + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \rho_{T-1}^{\alpha_{T-1}}(c_{T}(s_{T})) \right) \cdots \right) \right) \right\}$$

$$= \min_{\pi_{0},\dots,\pi_{T-2}} \left\{ c_{0}(s_{0},\pi_{0}(s_{0})) + \rho_{0}^{\alpha_{0}} \left(c_{1}(s_{1},\pi_{1}(s_{1})) + \rho_{1}^{\alpha_{1}} \left(c_{2}(s_{2},\pi_{2}(s_{2})) + \cdots \right) + \rho_{T-2}^{\alpha_{T-2}} \left(\min_{\pi_{T-1}} \left[c_{T-1}(s_{T-1},\pi_{T-1}(s_{T-1})) + \rho_{T-1}^{\alpha_{T-1}}(c_{T}(s_{T})) \right] \right) \cdots \right) \right) \right\}.$$

Let us focus now on the innermost optimization problem. By assumption the threshold conditional risk measure $\rho_{T-1}^{\alpha_{T-1}}$ is Markov, so we can rewrite the innermost optimization problem as follows:

(6.15)

$$\min_{\pi_{T-1}} \left\{ c_{T-1}(s_{T-1}, \pi_{T-1}(s_{T-1})) + \sigma_{T-1}(v_T, \overline{\alpha}_{T-1}, s_{T-1}, Q_{T-1}(s_{T-1}, \pi_{T-1}(s_{T-1}))) \right\}.$$

Letting $a_{T-1} = \pi_{T-1}(s_{T-1})$, we get the reformulation:

(6.16)
$$\min_{a_{T-1}} \left\{ c_{T-1}(s_{T-1}, a_{T-1}) + \sigma_{T-1}(v_T, \overline{\alpha}_{T-1}, s_{T-1}, Q_{T-1}(s_{T-1}, a_{T-1})) \right\}$$

Notice that by definition and the Markov property we can view the a_{T-1} in (6.16) as being an action available at state s_{T-1} , i.e. $a_{T-1} \in A_{T-1}(s_{T-1})$. In this way the problem becomes equivalent to the problem in (6.12) for t = T - 1 and its solution is given by (6.14) for t = T - 1.

Assumptions (i) and (ii) allow us to apply Theorem 15 and obtain that the function $\sigma_{T-1}(v_T, \overline{\alpha}_{T-1}, s, Q_{T-1}(s, \cdot))$ is lower semicontinuous. Assumption (iv) states that $c_{T-1}(s, \cdot)$ is lower semicontinuous too. Then the compactness of the set $A_{T-1}(s)$, which is given by assumption (v), guarantee us that problem (6.12) for t = T - 1 has an optimal solution for every $s \in S$. This optimal solution, denoted by $a_{T-1} = \hat{\pi}_{T-1}(s)$ is a measurable function of s (see [26, thm. 14.37]).

Since $c_T \in \mathcal{V}$, it follows from definition 7 that the function v_{T-1} is an element of \mathcal{V} . Therefore we can conclude that problem (6.10) is equivalent to the problem

(6.17)
$$\min_{\pi_0, \dots, \pi_{T-2}} \left\{ c_0(s_0, \pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1, \pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2, \pi_2(s_2)) + \cdots \right) \right) + \rho_{T-3}^{\alpha_{T-3}} \left(c_{T-2}(s_{T-2}, \pi_{T-2}(s_{T-2})) + \rho_{T-2}^{\alpha_{T-2}}(v_{T-1}(s_{T-1})) \right) \cdots \right) \right\} \right\}.$$

This is just a problem similar to (6.10) where the horizon has been decreased by one and the terminal cost equals $v_{T-1}(s_{T-1})$. We can easily check that problem (6.17) satisfies all the conditions of the statement of this theorem. Therefore proceeding recursively in this way for $t = T - 1, \ldots, 1$, we obtain the desired result.

In view of Theorem 16 we call the functions v_t , t = 0, ..., T - 1, the risk adjusted threshold value functions. These equations give a simple algorithm to solve finite horizon problems.

6.3. Optimality Equations Using the Post-Decision State Variable. In order to simplify the Bellman-type equations (6.11)–(6.14) we restate our risk-averse dynamic programming problem using post-decision state variables. The *post-decision state variable* is a powerful construct that represent the state of the system after we have made a decision but just before any new exogenous information has arrived. For a thorough exposition on the subject see [23, ch. 4].

Assume that we are under the same conditions as of Theorem 16. For any $t = 0, \ldots, T-1$ define the risk-adjusted value of being in state s immediately after taking decision $a \in A(s)$ by

(6.18)
$$v_t^a(s) = \sigma_t \left(v_{t+1}, \bar{\alpha}_t, s, Q_t(s, a) \right).$$

Then it is easy to see that the value function v_t defined in (6.12) satisfies

(6.19)
$$v_t(s) = \min_{a \in A_t(s)} \{ c_t(s, a) + v_t^a(s) \}$$

for all $s \in S$ and t = 0, ..., T - 1. We call v_t^a the risk-adjusted post-decision value function. From Theorem 16 we obtain the following post-decision state Bellman-type equations:

Theorem 17. Consider the risk-averse problem (6.10) together with all mathematical constructions and definitions given in this section. Assume that all the conditions of Theorem 16 are satisfied.

Then problem (6.10) has an optimal solution and its optimal value $v_0(s_0)$ is the solution of the following post-decision Bellman-type equations:

(6.20)
$$v_T(s) = c_T(s), \quad s \in \mathcal{S},$$

(6.21)
$$v_t(s) = \min_{a \in A_t(s)} \left\{ c_t(s, a) + v_t^a(s) \right\}, \ s \in \mathcal{S}, \ t = T - 1, \dots, 0,$$

where for $t = 0, \ldots, T - 1$ we have

(6.22)
$$v_t^a(s) = \sigma_t (v_{t+1}, \bar{\alpha}_t, s, Q_t(s, a)).$$

Moreover, an optimal Markov policy $\hat{\pi} = \{\hat{\pi}_0, \dots, \hat{\pi}_{T-1}\}$ exists and satisfies the equations:

(6.23)
$$\hat{\pi}_t(s) \in \operatorname*{arg\,min}_{a \in A_t(s)} \{ c_t(s, a) + v_t^a(s) \}, \ s \in \mathcal{S}, \ t = T - 1, \dots, 0.$$

Conversely, any measurable solution of equations (6.20)–(6.23) is an optimal Markov policy $\hat{\pi}$.

The changes in Theorem 17 seem to be minimal but are accompanied with an important shift in perspective. By focusing on the post-decision state variables our ADP algorithms use approximations to the post-decision value functions and thus "hide" the computationally expensive expectations with the effect of simplifying our numerical methods. This new outlook helps us in the contest against the curse of dimensionality that plagues dynamic programming problems. For an in-depth look at "pre-" and "post-" decision states and value functions in the context of dynamic programming see [23, ch. 4].

6.4. Evaluating Risk Only At The End. Suppose that we are not concerned with the risk incurred on the intermediate decision steps but we do care about the risk attained at the very last step of our finite horizon dynamic model. In some of our applications we have seen that this attitude towards risk is quite natural given that the cost function aggregates the demand with all the profits and losses at the terminal time T. An example of where this come into play are the time-lagged finite horizon problems where some of the most relevant random events are deferred to the last step, see [23, ch. 2].

We model this case by replacing the risk factor η in the threshold conditional risk measure by a time dependent risk factor η_t , $t = 0, \ldots, T - 1$. It is not difficult to see that even with this change in the risk factor all of our previous results hold. To strictly defer the risk evaluation to the very last step we simply have to require that $\eta_t = 0$ for all $t = 0, \ldots, T - 2$. With this in mind we obtain the following result.

Theorem 18. Consider the time lagged version described above of the risk-averse problem (6.10) together with all mathematical constructions and definitions given in this section. Assume that all the conditions of Theorem 16 are satisfied.

Then problem (6.10) has an optimal solution and its optimal value $v_0(s_0)$ is the solution of the following Bellman-type equations:

(6.24)
$$v_T(s_T) = c_T(s_T), \quad s_T \in \mathcal{S},$$

(6.25)
$$v_{T-1}(s_{T-1}) = \min_{a \in A_{T-1}(s_{T-1})} \left\{ c_{T-1}(s_{T-1}, a) + \sigma_{T-1} \left(v_T, \overline{\alpha}_{T-1}, s_{T-1}, Q_{T-1} \left(s_{T-1}, a \right) \right) \right\}, \ s_{T-1} \in \mathcal{S},$$

(6.26)
$$v_t(s_t) = \min_{a \in A_t(s_t)} \left\{ c_t(s_t, a) + \mathbb{E} \left[v_{t+1} \mid s_t, a \right] \right\}, \ s_t \in \mathcal{S}, \ t = T - 2, \dots, 0$$

Moreover, an optimal Markov policy $\hat{\pi} = {\hat{\pi}_0, \dots, \hat{\pi}_{T-1}}$ exists and satisfies the equations:

(6.27)
$$\hat{\pi}_{T-1}(s) \in \underset{a \in A_{T-1}(s)}{\operatorname{arg\,min}} \left\{ c_{T-1}(s,a) + \sigma_{T-1}(v_T, \overline{\alpha}_{T-1}, s, Q_{T-1}(s,a)) \right\}, \ s \in \mathcal{S},$$

(6.28)
$$\hat{\pi}_t(s) \in \underset{a \in A_t(s)}{\arg\min} \left\{ c_t(s, a) + \mathbb{E} \left[v_{t+1} \mid s, a \right] \right\}, \ s \in \mathcal{S}, \ t = T - 2, \dots, 0$$

Conversely, any measurable solution of equations (6.24)–(6.28) is an optimal Markov policy $\hat{\pi}$.

Theorem 18 states that we can solve this variant of the problem by applying the typical Bellman equations at all but the T-1 state where we need to take into consideration the desired threshold risk measure. Consider a new set of cost functions $\{\tilde{c}_t\}_{t=0}^{T-1}$ given by $\tilde{c}_t = c_t, t = 0, \ldots, T-2$ and

(6.29)
$$\tilde{c}_{T-1}(s,a) = c_{T-1}(s,a) + \sigma_{T-1}(v_T, \overline{\alpha}_{T-1}, s, Q_{T-1}(s,a)),$$

for all $(s, a) \in \operatorname{graph}(A)$. Then we can see that another way to deal with the time lagged problem is to solve a classical risk-neutral dynamic program with the new cost functions $\tilde{c}_t, t = 0, \ldots, T - 1$ and every other component equal to the time lagged problem discussed in Theorem 18. This method has the advantage of reducing the problem to a regular dynamic program to which we can apply a wide array of methods to obtain or approximate its solution (see [23]).

7. STATIC-THRESHOLD DYNAMIC RISK MEASURES

Consider now a threshold sequence $\boldsymbol{\alpha} = \{\alpha_t\}_{t=0}^{T-1}$ such that for every $t = 0, \ldots, T-1$, the threshold function α_t is constant. This threshold sequence is *static* in the sense that at any given moment the threshold does not depend on either the state or the actions taken. *Static-threshold* sequences can be represented as sequences $\boldsymbol{\alpha} \in \mathbb{R}^T_+$, where each $\boldsymbol{\alpha}$ can be seen as a series of goals to meet along our dynamic optimization process.

In this section we focus on threshold dynamic risk measures with static-threshold sequences. We call these, *static-threshold dynamic risk measures*. A fixed static-threshold sequence yields a special, and in essence simpler, case of the optimization problem (6.10) and as such, we can apply all the techniques already discussed in previous sections to it. The interesting case is when we are presented with a range of static-threshold sequences and we must select the one that is "best" for our problem.

To this end we take a robust optimization point of view and seek to choose the best policy subjected to its worst possible static-threshold sequence. In more formal words, we want to solve

(7.1)
$$\min_{\pi \in \Pi} \max_{\boldsymbol{\alpha} \in X} R(s_0, \pi, \boldsymbol{\alpha})$$

where Π denotes the set of admissible policies, $X \subseteq \mathbb{R}^T_+$ is a convex set, and

$$R(s_0, \pi, \alpha) = c_0(s_0, \pi_0(s_0)) + \rho_0^{\alpha_0} \left(c_1(s_1, \pi_1(s_1)) + \rho_1^{\alpha_1} \left(c_2(s_2, \pi_2(s_2)) + \dots + \rho_{T-2}^{\alpha_{T-2}} \left(c_{T-1}(s_{T-1}, \pi_{T-1}(s_{T-1})) + \rho_T^{\alpha_T} \right) \right) \right)$$

with $\pi = {\pi_t}_{t=0}^{T-1}$ and $\alpha = {\alpha_t}_{t=0}^{T-1}$. Problem (7.1) is a difficult min-max non-convex optimization problem and its analysis lies outside the scope of this paper. Instead, we focus on a special case where, thanks to the properties of threshold dynamic risk measures, the optimization of (7.1) follows simple rules.

7.1. Minimum Static-Threshold Sequences. Suppose that the convex set of static-thresholds, $X \subseteq \mathbb{R}^T_+$, contains a minimum element. That is, an element $\boldsymbol{\alpha}^* \in X$ such that $\boldsymbol{\alpha}^* \leq \boldsymbol{\alpha}$ for every $\boldsymbol{\alpha} \in X$, where \leq denotes the componentwise partial order on \mathbb{R}^T . Clearly, such a minimum element $\boldsymbol{\alpha}^*$ is unique. In this case, property (*DR4*) of Theorem 10 gives that $\rho_{0,T}^{\boldsymbol{\alpha}^*}(Z) \geq \rho_{0,T}^{\boldsymbol{\alpha}}(Z)$, for every $\boldsymbol{\alpha} \in X$ and every $Z \in \mathcal{Z}_{0,T}$. Therefore we can conclude that for any policy $\pi \in \Pi$:

$$\max_{\boldsymbol{\alpha} \in \boldsymbol{X}} R(s_0, \pi, \boldsymbol{\alpha}) = R(s_0, \pi, \boldsymbol{\alpha}^*).$$

Thus, the solution to problem (7.1) is obtained by solving

(7.2)
$$\min_{\pi \in \Pi} R(s_0, \pi, \boldsymbol{\alpha}^*),$$

and this can be done applying the techniques discussed in previous sections.

This is summarized in the following theorem:

Theorem 19. Let $\left\{\rho_{t,T}^{\boldsymbol{\alpha}}\right\}_{t=0}^{T}$ be a static-threshold dynamic risk measure and $X \subseteq \mathbb{R}_{+}^{T}$ be a convex set of static-thresholds with a minimum element $\boldsymbol{\alpha}^{*} \in X$. Then the solution of the min-max optimization problem (7.1) is given by

$$\min_{\pi \in \Pi} R(s_0, \pi, \boldsymbol{\alpha}^*).$$

The following are some brief examples where Theorem 19 is applied.

Example 8. Consider the case where we have fixed thresholds $0 \leq \beta_t \leq \gamma_t, t = 0, \ldots, T-1$, and $X = \left\{ \boldsymbol{\alpha} = \{\alpha_t\}_{t=0}^{T-1} \mid \beta_t \leq \alpha_t \leq \gamma_t, t = 0, \ldots, T-1 \right\}$. The set X forms a convex multidimensional box and has as minimum element the sequence $\boldsymbol{\beta} = \{\beta_t\}_{t=0}^{T-1}$. Therefore by Theorem 19, problem (7.1) is given by

$$\min_{\pi \in \Pi} R(s_0, \pi, \beta)$$

Example 9. Consider the multidimensional box X defined in example 8, let \widetilde{X} be the set of nondecreasing sequences in X, and suppose that \widetilde{X} is not empty. Then \widetilde{X} is convex and has as minimum element the sequence $\widetilde{\alpha} = \{\widetilde{\alpha}_t\}_{t=0}^{T-1}$, where $\widetilde{\alpha}_0 = \beta_0$ and $\widetilde{\alpha}_t = \min \{\alpha \in [\beta_t, \gamma_t] \mid \widetilde{\alpha}_{t-1} \leq \alpha\}, t = 1, \ldots, T-1$. Therefore we can apply Theorem 19 to solve problem (7.1) over the set of static-thresholds \widetilde{X} .

Example 10. Similar to example 9, consider the multidimensional box X defined in example 8 and let \hat{X} be the set of nonincreasing sequences in X. As before, suppose that \hat{X} is not empty. Then \hat{X} is convex and has as minimum element the sequence $\hat{\alpha} = \{\hat{\alpha}_t\}_{t=0}^{T-1}$, where $\hat{\alpha}_{T-1} = \beta_{T-1}$ and $\hat{\alpha}_t = \min\{\alpha \in [\beta_t, \gamma_t] \mid \alpha \leq \hat{\alpha}_{t+1}\}, t = T -$

2,...,0. Therefore we can apply Theorem 19 to solve problem (7.1) over the set of static-thresholds \hat{X} .

Theorem 19 encompasses all the applications for static-threshold sequences that we have so far encountered in our research. As an illustration, the nondecreasing sequences of example 9 can be seen as a relaxation of the risk-aversion towards later stages of the time horizon, while the nonincreasing sequences of example 10 can be interpreted as a tightening of the risk-aversion. Both of these attitudes towards risk are natural in our applications to energy systems and markets where we apply them to obtain new risk-averse policies.

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Appendix A

Proof of Theorem 1. Let $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, let $\alpha \in \mathcal{Z}$ be a threshold function and consider the nonnegative operator $[\cdot]_+$.

(N0) Closeness: Let $Z \in \mathcal{Z}$, then by the Minkowski inequality

$$||[Z - \alpha]_+||_p \le || |Z| + |\alpha| ||_p \le ||Z||_p + ||\alpha||_p < \infty.$$

Therefore $[Z - \alpha]_+ \in \mathcal{Z}$.

(N1) Convexity: Let $Z, W \in \mathcal{Z}$ and $t \in [0, 1]$. Then for every $\omega \in \Omega$

$$[tZ(\omega) + (1-t)W(\omega)]_{+} \le t[Z(\omega)]_{+} + (1-t)[W(\omega)]_{+}$$

Therefore $[tZ + (1-t)W]_+ \leq t[Z]_+ + (1-t)[W]_+$.

Properties (N2)–(N5) follow similarly by application to single elements $\omega \in \Omega$. \Box

Proof of Theorem 2. Let $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P), Z \in \mathcal{Z}, \alpha \in \mathcal{Z}$ be a threshold function, and let $\mathfrak{B} \in \mathcal{F}$ be a random event of the probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Suppose that $\omega \in \mathfrak{B}$, then $\mathbb{1}_{\mathfrak{B}}(\omega) = 1$ and

(A.1)
$$[\mathbb{1}_{\mathfrak{B}}(\omega)Z(\omega) - \alpha(\omega)]_{+} = [Z(\omega) - \alpha(\omega)]_{+} = \mathbb{1}_{\mathfrak{B}}(\omega)[Z(\omega) - \alpha(\omega)]_{+}.$$

On the other hand, if $\omega \notin \mathfrak{B}$, then $\mathbb{1}_{\mathfrak{B}}(\omega) = 0$ and

(A.2)
$$[\mathbb{1}_{\mathfrak{B}}(\omega)Z(\omega) - \alpha(\omega)]_{+} = [0 - \alpha(\omega)]_{+} = 0 = \mathbb{1}_{\mathfrak{B}}(\omega)[Z(\omega) - \alpha(\omega)]_{+},$$

where the second equality is justify by the fact that the threshold function α is nonnegative. Clearly (A.1) and (A.2) imply that $[\mathbb{1}_{\mathfrak{B}}Z - \alpha]_{+} = \mathbb{1}_{\mathfrak{B}}[Z - \alpha]_{+}$.

Proof of Theorem 3. Let $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P), Z \in \mathcal{Z}$, and $\alpha \in \mathcal{Z}$ be a threshold function. Consider the optimization problem on random variables:

(A.3)
$$\begin{array}{c} \min_{X} X \\ \text{s.t. } X \succeq Z - \alpha \\ X \succeq 0, X \in \mathcal{Z} \end{array}$$

Then for every $\omega \in \Omega$ the random variable $[Z - \alpha]_+$ satisfies that $[Z - \alpha]_+(\omega) \ge Z(\omega) - \alpha(\omega)$ and $[Z - \alpha]_+(\omega) \ge 0$. Therefore $[Z - \alpha]_+$ is a feasible solution to problem (A.3).

Suppose now that X is a feasible solution to problem (A.3), then a.e. $\omega \in \Omega$

$$X(\omega) \ge \max\{Z(\omega) - \alpha(\omega), 0\} = [Z - \alpha]_+(\omega)$$

Therefore $X \succeq [Z - \alpha]_+$ and we conclude that $[Z - \alpha]_+$ is an optimal solution to (A.3).

Proof of Theorem 4. Let $p \in [1, +\infty)$, $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, and let $\varrho : \mathcal{Z} \to \mathbb{R}$ and $\vartheta : \mathcal{Z} \to \mathbb{R}$ be real-valued, lower semicontinuous coherent risk measures. Let $\eta > 0$, $\alpha \in \mathcal{Z}$ be a threshold function and consider the risk measure of threshold α given by

$$\rho^{\alpha}(Z) = \varrho(Z) + \eta \,\vartheta([Z - \alpha]_+),$$

for every $Z \in \mathcal{Z}$.

Properties (T1) and (T2) are clearly inherited from the properties of coherent risk measures.

(T3) Translation equivariance: Let $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then

(A.4)

$$\rho^{\alpha}(Z+a) = \varrho(Z+a) + \eta \,\vartheta([Z+a-\alpha]_{+})$$

$$= \varrho(Z) + a + \eta \,\vartheta([Z-(\alpha-a)]_{+})$$

$$= \rho^{\alpha-a}(Z) + a,$$

where in the second inequality we used the translation equivariance property of coherent risk measures.

(T4) Positive homogeneity: Let t > 0 and $Z \in \mathbb{Z}$, then

(A.5)

$$\rho^{\alpha}(tZ) = \varrho(tZ) + \eta \,\vartheta([tZ - \alpha]_{+})$$

$$= \varrho(tZ) + \eta \,\vartheta(t[Z - \alpha/t]_{+})$$

$$= t \left(\varrho(Z) + \eta \,\vartheta([Z - \alpha/t]_{+})\right)$$

$$= t\rho^{\alpha/t}(Z),$$

where we used property (N5) from Theorem 1 to justify the second equality in (A.5).

Observe now that by (N0) of Theorem 1 and the fact that both ρ and ϑ are real-valued, the function ρ^{α} is real-valued and therefore a risk measure. Clearly ρ^{α} inherits the lower semicontinuity from ρ and ϑ . Being real-valued, convex, and lower semicontinuous everywhere imply that ρ^{α} is subdifferentiable on \mathcal{Z} .

Proof of Theorem 5.Let $p \in [1, +\infty)$, $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, and let $\varrho : \mathcal{Z} \to \mathbb{R}$ and $\vartheta: \mathcal{Z} \to \mathbb{R}$ be real-valued, lower semicontinuous coherent risk measures. Let $\eta > 0$, $\alpha, \beta \in \mathcal{Z}$ be threshold functions.

(C1) Convexity of thresholds: For every $Z \in \mathcal{Z}$ and every $\lambda \in [0, 1]$:

$$\begin{split} \rho^{\lambda\alpha+(1-\lambda)\beta}(Z) &= \varrho(Z) + \eta \,\vartheta([Z - (\lambda\alpha + (1-\lambda)\beta)]_+) \\ &= \varrho \left(\lambda Z + (1-\lambda)Z\right) + \eta \,\vartheta\left([\lambda(Z-\alpha) + (1-\lambda)(Z-\beta)]_+\right) \\ &\leq \varrho \left(\lambda Z + (1-\lambda)Z\right) + \eta \,\vartheta\left([\lambda(Z-\alpha)]_+ + [(1-\lambda)(Z-\beta)]_+\right) \\ &\leq \lambda \left(\varrho(Z) + \eta \,\vartheta([Z-\alpha]_+)\right) + (1-\lambda) \left(\varrho(Z) + \eta \,\vartheta([Z-\beta]_+)\right) \\ &= \lambda \rho^{\alpha}(Z) + (1-\lambda)\rho^{\beta}(Z), \end{split}$$

where the first inequality is obtained from property (N4) of Theorem 1 and the monotonicity of coherent risk measures.

(C2) Monotonicity of Thresholds: Let $\alpha, \beta \in \mathbb{Z}$ be threshold functions such that $\alpha \preceq \beta$, then for every $Z \in \mathcal{Z}$ we have that $Z - \alpha \succeq Z - \beta$. Therefore by property (N2) from Theorem 1 we understand that $[Z - \alpha]_+ \succeq [Z - \beta]_+$. Then by the monotonicity of coherent risk measures we see that $\rho^{\alpha}(Z) \ge \rho^{\beta}(Z)$.

Proof of Theorem 6. The first representation theorem of threshold risk measures is a simple consequence of basic duality result for convex risk measures presented in [32, thm. 6.4].

Proof of Theorem 7. Let $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$ and $\mathcal{Z}^* = \mathcal{L}_q(\Omega, \mathcal{F}, P)$ be a conjugate pair of spaces. Let $\eta > 0, \alpha \in \mathcal{Z}$ be a threshold function, and let $\rho: \mathcal{Z} \to \mathbb{R}, \vartheta: \mathcal{Z} \to \mathbb{R}$ be real-valued, lower semicontinuous coherent risk measures. Consider the threshold risk measure given by

$$\rho^{\alpha}(Z) = \varrho(Z) + \eta \,\vartheta([Z - \alpha]_+),$$

for all $Z \in \mathcal{Z}$.

The representation theorem of coherent risk measures [32, thm. 6.4] imply that for every $Z \in \mathcal{Z}, \rho^{\alpha}(Z)$ is the optimal solution to

(A.6)
$$\sup_{\substack{\mu,\zeta\\ \text{s.t. } \mu \in \partial \varrho(0), \ \zeta \in \partial \vartheta(0),}} \langle \mu, Z \rangle + \eta \langle \zeta, [Z - \alpha]_+ \rangle$$

where the subdifferentials $\partial \rho(0)$ and $\partial \vartheta(0)$ are closed convex sets of probability density functions on \mathcal{Z} with respect to the reference probability P. Then applying the linear representation of the nonnegative operator (Theorem 3) to the random variable $[Z-\alpha]_+$ we obtain that $\rho^{\alpha}(Z)$ is the optimal solution to the following optimization problem:

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(A.7)
$$\begin{aligned} \sup_{\mu,\zeta} \ \min_{X} \ \langle \mu, Z \rangle + \eta \left\langle \zeta, X \right\rangle \\ \text{s.t. } X \succeq Z - \alpha \\ X \succeq 0, \ X \in \mathcal{Z} \\ \mu \in \partial \varrho(0), \ \zeta \in \partial \vartheta(0). \end{aligned}$$

Proof of Theorem 8. Let $\rho_t^{\alpha_t} : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ be a one-step threshold conditional risk measure with threshold function $\alpha_t \in \mathcal{Z}_{t+1}, t = 0, \ldots, T-1$. (*T0'*): Notice that by definition $\rho_0^{\alpha_0}$ is a threshold risk measure and as such it is real-valued, lower semicontinuous and subdifferentiable.

Properties (T1')-(T2') are clearly inherited from the properties of one-step conditional coherent risk measures. Properties (T3')-(T4') are proved in a fashion similar to the proofs of (T3')-(T4) from Theorem 4.

Proof of Theorem 9. Properties (C1')-(C2') are proved in a fashion similar to the proofs of (C1')–(C2') from Theorem 5.